Theory by Chapter

Chapter P

Absolute Value: The absolute value of a real number a is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Note: You can think of $|a|$ as the distance of the point a from the origin on the number line.

***n*th Root:** (This definition has two parts.)

- If n is an even positive integer and $b \geq 0$, then the principal n^{th} root of b, $\sqrt[n]{b}$, is the nonnegative real number, a, such that $a^n = b$; and
- If n is an odd positive integer, then the n^{th} root of b, $\sqrt[n]{b}$, is the real number, a, such that $a^n = b$.

Note: When n is an even, the negative n^{th} root of b is written $-\sqrt[n]{b}$.

Polynomial in One Variable: The standard form of a polynomial of degree n in the variable x is

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and n is a nonnegative integer.

Rational Expression: A rational expression is an expression of the form $\dfrac{P}{Q}$ where P and Q are polynomials and $Q \neq 0$.

Complex Number: A complex number is a number of the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$.

Chapter 1

Quadratic Equation: A quadratic equation in the variable x is an equation that can be written in the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$.

Zero Product Principle: If *A* and *B* are algebraic expressions such that $AB = 0$, then $A = 0$ or $B = 0$.

Quadratic Formula: If $ax^2 + bx + c = 0$, $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof:

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac + b^2}{4a^2}}$$
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Direct Variation: For variables x and y, y varies directly as x if and only if $y = kx$ where k is a constant.

Inverse Variation: For variables x and y, y varies inversely as x if and only if $y = \dfrac{k}{x}$ where k is a constant.

Chapter 2

Distance Formula: The distance between the points (x_1, y_1) and (x_2, y_2) is

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

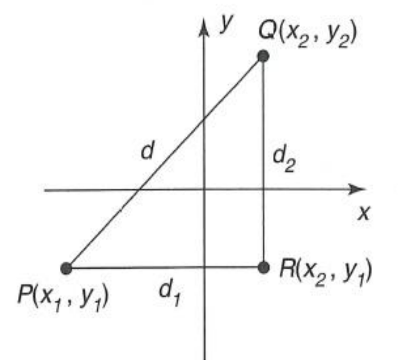
Proof: The points $P(x_1, y_1)$, $Q(x_2, y_2)$, and $R(x_2, y_1)$ form a right triangle. (See Figure.)
The distance between P and R is $d_1 = x_2 - x_1$ and the distance between Q and R is $d_1 = y_2 - y_1$.
We can apply the Pythagorean Theorem as follows:

$$d^2 = d_1^2 + d_2^2$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Since distance is always nonnegative,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Function: A function is a set of ordered pairs in which no two ordered pairs have the same first coordinate and different second coordinates.

Note: A function is a relation that pairs each element in the domain with a distinct element in the range.

One-to-One Function: A function, f, is a one-to-one function if and only if $f(a) = f(b)$ implies $a = b$.

Linear Function: A linear function of x is a function that can be represented by an equation of the form $f(x) = ax + b$ where a and b are real numbers and $a \neq 0$.

Slope: The slope m of the line passing through the points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Quadratic Function: A quadratic function of x is a function that can be represented by an equation of the form $f(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$.

Chapter 3

Remainder Theorem: If a polynomial $P(x)$ is divided by $x - c$, the remainder is $P(c)$.

Proof: The division of a polynomial $P(x)$ by $x - c$ results in a quotient $Q(x)$ and a constant remainder r such that

$$P(x) = (x - c) \cdot Q(x) + r$$

Since the identity holds for all real values of *x*, it must hold when *x* = *c*. Consequently,

$$P(c) = (c - c) \cdot Q(c) + r$$

$$P(c) = 0 \cdot Q(c) + r$$

$$P(c) = r$$

Factor Theorem: A polynomial $P(x)$ has a factor $x - c$ if and only if $P(c) = 0$.

Proof: (Part 1: if $P(x)$ has a factor $x - c$ then $P(c) = 0$.)

Let $x - c$ be a factor of $P(x)$.
By the definition of factor,

$$P(x) = (x - c) \cdot Q(x)$$

Since the identity holds for all real values of x, it must hold for $x = c$.
Consequently,

$$P(c) = (c - c) \cdot Q(c)$$

$$P(c) = 0 \cdot Q(c)$$

$$P(c) = 0$$

(Part 2: if $P(c) = 0$ then $P(x)$ has a factor $x - c$.)

Let c be a constant such that $P(c) = 0$.
By the Remainder Theorem,

$$P(x) = (x - c) \cdot Q(x) + P(c)$$

But $P(c) = 0$, so $\quad P(x) = (x - c) \cdot Q(x) + 0$

$$P(x) = (x - c) \cdot Q(x)$$

Thus, by definition of factor, $x - c$ is a factor of $P(x)$.

Chapter 3 continued

Intermediate Value Theorem: If P is a polynomial function such that $P(a) \neq P(b)$ for $a < b$, then P takes on every value between $P(a)$ and $P(b)$ in the interval $[a, b]$.

Rational Zero Theorem: If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients with $a_n \neq 0$ and $\frac{p}{q}$ is a rational zero in simplest form of P, then

1. p is a factor of the constant term a_0, and
2. q is a factor of the leading coefficient a_n.

Fundamental Theorem of Algebra: If P is polynomial of degree $n \geq 1$ with complex coefficients, then P has at least one zero among the complex numbers.

Linear Factor Theorem: If P is a polynomial such that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $n \geq 1$ and $a_n \neq 0$, then P is factorable into exactly n linear factors

$$P(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where c_1, c_2,…, c_n are complex numbers.

Chapter 4

Inverse Function: If the ordered pairs of a function g are the ordered pairs of a function f with the order of the coordinates reversed, then g is the *inverse function* of f.

Exponential Function: The *exponential function* with base b is defined by

$$f(x) = b^x$$

where $b > 0$, $b \neq 1$, and x is a real number.

Logarithmic Function: If $x > 0$ and b is a positive constant ($b \neq 1$), then

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

4.4 Properties of Logarithms (not on exam)

1. *Product property* $\quad \log_b(MN) = \log_b M + \log_b N$
2. *Quotient property* $\quad \log_b \frac{M}{N} = \log_b M - \log_b N$
3. *Power property* $\quad \log_b(M^p) = p \log_b M$

Change of Base Formula: If x, a, and b are positive real numbers with a ≠ 1 and b ≠ 1, then

$$\log_b x = \frac{\log_a x}{\log_a b}$$