1. Prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

2. Prove that for all $x, y \in \mathbb{Z}$, x and y are odd iff xy is odd.

3. Let f be differentiable and $f'(x) < 0$ for all x. Prove that f is decreasing. (Don't forget the Mean Value Theorem.)

4. Prove that there is no largest integer.

5. Prove that for all $x \in \mathbb{Z}^{>}$ (aka non-negative integers), there exists $q, r \in \mathbb{Z}$ such that $x = 3q + r$ where $0 \leq r < 3$.

6. Prove that the square of any integer is of the form $3k$ or $3k + 1$ (aka prove that if $x \in \mathbb{Z}$, then there exists $k \in \mathbb{Z}$ such that $x^2 = 3k$ or $x^2 = 3k + 1$). (Use the previous result.)

7. Prove that if $m|a$ and $n|b$, then $mn|ab$.

8. Prove that if $a + b > 100$, then either $a > 50$ or $b > 50$.

9. Prove $\sum_{i=1}^{n} (2i - 1) = n^2$.

10. Prove that if $x \in \mathbb{Z}$, then $x^2 + x$ is even.

1) Prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (Induction)

Let $k=1$ $\sum_{i=1}^{1} i^2 = 1^2 = \frac{1(2)(3)}{6}$ ✓

S true for $n=k-1$, then $\sum_{i=1}^{k-1} i^2 = \frac{(k-1)(k)(\overset{(2k-1)}{2k-2+1})}{6}$

Show true for $n=k$

$$\sum_{i=1}^{k} i^2 = \sum_{i=1}^{k-1} i^2 + k^2 = \frac{(k-1)(k)(2k-1)}{6} + k^2$$

$$= \frac{k}{6}\left((k-1)(2k-1) + 6k\right)$$

$$= \frac{k}{6}\left(2k^2 - 3k + 1 + 6k\right) = \frac{k}{6}(2k^2+3k+1)$$

$$= \frac{k}{6}(k+1)(2k+1) = \frac{k(k+1)(2k+1)}{6}$$

$\therefore$ the rule holds.

2) Let $x,y \in \mathbb{Z}$ prove x,y both odd iff xy odd

($\Rightarrow$) Let x,y both be odd $\Rightarrow \exists\, m,n \in \mathbb{Z} \ni$ (Direct)

$x = 2m+1$ and $y = 2n+1$

$$\Rightarrow xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1$$

$$= 2(2mn+m+n)+1$$

Since $m,n \in \mathbb{Z} \Rightarrow 2mn+m+n \in \mathbb{Z}$

$\therefore$ xy is odd

($\Leftarrow$) Let xy be odd $\Rightarrow \exists\, k \in \mathbb{Z} \ni xy = 2k+1$

S x and/or y is even (Contrapositive) + Cases

Case 1 one is odd + one is even

WOLOG (without Loss of generality)

let x be odd and y be even (Direct)

$\Rightarrow \exists\ m, n \in \mathbb{Z} \ni \ x = 2m+1$ and $y = 2n$

$\Rightarrow xy = (2m+1)(2n) = 2(2mn+n)$

$m, n \in \mathbb{Z} \Rightarrow 2mn+n \in \mathbb{Z}$

$\Rightarrow xy$ is even $\Rightarrow \Leftarrow$

$\therefore$ ~~xxxxx~~ it can't be that one is even + one is odd

Case 2 both are even

$\Rightarrow \exists\ \cancel{m}, n \in \mathbb{Z} \ni \ x = 2m$ and $y = 2n$

(Direct)

$\Rightarrow xy = 2m(2n) = 2(2mn)$

Since $m, n \in \mathbb{Z} \Rightarrow 2mn \in \mathbb{Z}$

$\therefore$ xy is even $\Rightarrow \Leftarrow$

$\therefore$ it can't be that both x, y are even

but at least one is even $\Rightarrow \Leftarrow$

$\therefore$ x and y must both be odd

3) recall MVT: Let f be cont on $[a,b]$ + diff on (a,b) then $\exists\ c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Prove if f is diff + $f'(x) < 0\ \forall x$ then f is decreasing

(Direct)

S $x < y$ show $f(x) > f(y)$

by MVT $\exists\ z \in (x,y) \ni f'(z) = \dfrac{f(y)-f(x)}{y-x}$

(since f is diff $\Rightarrow f$ is cont).

Since $f'(x) < 0\ \forall x \Rightarrow f'(z) = \dfrac{f(y)-f(x)}{y-x} < 0$

+ since $x < y \Rightarrow y - x > 0$, then $f(y) - f(x) < 0$

$\Rightarrow f(y) < f(x)$ $\therefore$ f is decreasing

4) Prove that there is no largest integer

$S \exists k \in \mathbb{Z} \ni k > n \ \forall n \in \mathbb{Z}$ (Contradiction)

Consider $k+1$, since $k \in \mathbb{Z}$, $k+1 \in \mathbb{Z}$

and since $k+1 > k \Rightarrow k \not> n \ \forall n \in \mathbb{Z}$

$\therefore \nexists$ a largest integer

5) Let $x \in \mathbb{Z}^>$, prove $\exists q, r \in \mathbb{Z} \ni x = 3q + r, 0 \leq r < 3$

(Induction)

$\cancel{k}x = 1$ let $q = 0$, $r = 1$

then $x = 3q + r$

$S \exists q, r \in \mathbb{Z} \ni x = n = 3q + r$ (w/ $0 \leq r < 3$)

Consider $x = n+1$

$n+1 = (3q + r) + 1$

if $r = 0$ then set $q' = q$ and $r' = 1$ (cases)

and $n+1 = 3q' + r'$

if $r = 1$ then set $q' = q$ and $r' = 2$

and $n+1 = 3q' + r'$

if $r = 2$ then set $q' = q+1$ and $r' = 0$

and $n+1 = 3q' + r'$

$\therefore \exists q, r \in \mathbb{Z} \ni$

$x = 3q + r$ w/ $0 \leq r < 3$

6) given for any $x \in \mathbb{Z}^{>}$ $\exists q, r \in \mathbb{Z} \ni x = 3q + r$

show that for any $n \in \mathbb{Z}$ $n^2 = 3k$ or $n^2 = 3k+1$
for some $k \in \mathbb{Z}$

Since $n \in \mathbb{Z} \Rightarrow n^2 \in \mathbb{Z}^{>}$ if $n \neq 0$

(cases)

if $n = 0$ then $n^2 = 0 = 3(0)$ Let $k = 0$

if $n \neq 0$ then $n^2 \in \mathbb{Z}^{>}$ and $\exists q, r \in \mathbb{Z}$, $0 \leq r \leq 2$

where $|n| = 3q + r$

$\Rightarrow n^2 = (|n|)^2 = (3q+r)^2 = 9q^2 + 6qr + r^2$

if $r = 0$ then $n^2 = 9q^2 = 3(3q^2)$ (cases)

Since $q \in \mathbb{Z} \Rightarrow 3q^2 \in \mathbb{Z}$

if $r = 1$ then $n^2 = 9q^2 + 6q + 1$

$= 3(3q^2 + 2q) + 1$

Since $q \in \mathbb{Z} \Rightarrow 3q^2 + 2q \in \mathbb{Z}$

if $r = 2$ then $n^2 = 9q^2 + 12q + 4$

$= (9q^2 + 12q + 3) + 1$

$= 3(3q^2 + 4q + 1) + 1$

Since $q \in \mathbb{Z} \Rightarrow 3q^2 + 4q + 1 \in \mathbb{Z}$

7) Prove if $m | a$ and $n | b$ then $mn | ab$

Let $m | a$ and $n | b$ $\Rightarrow \exists x, y \in \mathbb{Z} \ni a = mx$, $b = ny$

$\Rightarrow ab = mx(ny) = mn(xy)$ since $x, y \in \mathbb{Z} \Rightarrow xy \in \mathbb{Z}$

$\Rightarrow \exists w = xy \in \mathbb{Z} \ni ab = mnw \Rightarrow mn | ab$

8) if $a+b>100$ show $a>50$ or $b>50$

S $a \le 50$ and $b \le 50$ (Contrapositive)

$\Rightarrow a+b \le 100 \Rightarrow \Leftarrow$

$\therefore$ either $a>50$ or $b>50$

ALTERNATE PF:

Let $a+b>100$ (CASES)

either $a>50$ (+the result holds)

or $a \le 50 \Rightarrow 50 \ge a$

$50+b \ge a+b > 100$

$\Rightarrow 50+b>100$

$\Rightarrow b>50$ (+the result holds).

9) Prove $\sum_{i=1}^{n}(2i-1)=n^2$ (Induction)

$n=1$ $\sum_{i=1}^{1}(2i-1)=2-1=1=1^2$ ✓

S holds for $n=k-1$ $\Rightarrow \sum_{k=1}^{k-1}(2i-1)=(k-1)^2=k^2-2k+1$

show true for $n=k$

$\sum_{k=1}^{k}(2i-1)=\sum_{i=1}^{k-1}(2i-1)+2k-1=k^2-2k+1+2k-1$

$=k^2$ result holds

10) if $x \in \mathbb{Z}$ show x^2+x is even

if x is even $\Rightarrow \exists k \in \mathbb{Z} \ni x=2k$ (cases)

$x^2+x=(2k)^2+2k=2(2k^2+k)$ since $k \in \mathbb{Z} \Rightarrow 2k^2+k \in \mathbb{Z}$

$\Rightarrow x^2+x$ is even

if x is odd $\Rightarrow \exists k \in \mathbb{Z} \ni x=2(k)+1$

$x^2+x=(2k+1)^2+2k+1=4k^2+4k+1+2k+1$

$=4k^2+6k+2=2(2k^2+3k+1)$ since $k \in \mathbb{Z} \Rightarrow 2k^2+3k+1 \in \mathbb{Z}$

$\Rightarrow x^2+x$ is even