1. Let X be a continuous uniform random variable on the interval (a, b), where the density function is $f(x) = \frac{1}{b-a}$. Prove that this is a valid density function and that the mean, $E[X]$, is the midpoint between a and b. (Review definitions from Density Functions Handout.)

2. Disprove the statement that if $f(x)$ is continuous at x, then it is differentiable at x.

3. Prove that there exists a function f, such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$.

4. If $n \neq 0$, $a \in \mathbb{Z}$, then n divides a, denoted $n|a$, if $\exists m \in \mathbb{Z}$ such that $a = nm$. Prove that if $n|a$ and $a|b$, then $n|b$.

5. If $n \in \mathbb{Z}$ and $n > 0$ and $4^n - 1$ is prime, then n is odd.

6. Let $n \in \mathbb{Z}$ and $n > 0$, prove $\frac{d}{dx}x^n = nx^{n-1}$.

7. Prove that if x is even, then x^2 is even.

8. Prove that x is even iff $x + 1$ is odd.

9. Prove that for all $x \in \mathbb{Z}$, $x^2 + x$ is even.

10. Prove that for any rational x and any irrational y, $x + y$ is irrational.

11. Let $a_i \in \mathbb{R}$ for $i = 1, \ldots, n$. If $\prod_{i=1}^n a_i = a_1 a_2 \ldots a_n = 0$, then prove $\exists\ k \in \{1, 2, \ldots, n\}$ such that $a_k = 0$.

12. Prove that for all x there exists y such that $y^2 > x$.

1) if $f(x) = \frac{1}{b-a}$ on $x \varepsilon (a,b)$

$\Rightarrow b > a \Rightarrow b-a > 0 \Rightarrow \frac{1}{b-a} > 0$

$\int_a^b f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a} x \Big|_a^b = \frac{1}{b-a}(b-a) = 1$

$\therefore$ a valid density function

$E[x] = \int_a^b x\, f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{x^2}{2} \Big|_a^b$

$= \frac{b^2-a^2}{2(b-a)} = \frac{(\cancel{b-a})(b+a)}{2(\cancel{b-a})} = \frac{a+b}{2}$ = mid pt b/w a+b.

2) Let $f(x) = |x|$, which is continuous $\forall x \varepsilon \mathbb{R}$

Consider $x=0$ $f'(0)$ DNE $\therefore$ if f is cont. then f is diff does not hold.

3) Let $f(x) = e^x$

$\Rightarrow f'(x) = e^x \Rightarrow \exists$ a function $\ni$ $f(x) = f'(x)$ $\forall x$

4) Let $n | a$ and $a | b$

$\Rightarrow \exists\ m_1, m_2 \varepsilon \mathbb{Z} \ni a = m_1 n$ and $b = m_2 a$

$\Rightarrow b = m_2(m_1 n) = (m_2 m_1) n$

Since $m_1, m_2 \varepsilon \mathbb{Z} \Rightarrow m_2 m_1 \varepsilon \mathbb{Z} \Rightarrow \exists\ m_3 = m_2 m_1 \varepsilon \mathbb{Z}$

$\ni b = m_3 n \therefore n | b$

5) Let $n \varepsilon \mathbb{Z}$, $n > 0$ and let $4^n - 1$ be prime.

Suppose n is even $\Rightarrow \exists\ k \varepsilon \mathbb{Z} \ni n = 2k$

$\Rightarrow 4^n - 1 = 4^{2k} - 1 = (4^k)^2 - 1 = (4^k - 1)(4^k + 1)$

Since $n > 0 \Rightarrow k > 0$ and since $k \in \mathbb{Z} \Rightarrow k \geq 1$

Since $f(x) = 4^x$ is an increasing function

$$\Rightarrow \quad 4^k - 1 \geq 4^1 - 1 = 3$$

$$\text{and} \quad 4^k + 1 > 4^1 + 1 = 5$$

Since $4^k \pm 1 \in \mathbb{Z}$ and $4^k \pm 1 > 1$

$\Rightarrow 4^n - 1$ has factors other than 1 and $4^n - 1$

$\Rightarrow 4^n - 1$ is not prime $\Rightarrow\Leftarrow$

$\therefore$ n is not even $\Rightarrow$ n is odd

6) prove $\frac{d}{dx} x^n = n x^{n-1}$ for $n \in \mathbb{Z}^+$

$n = 1$ $\quad \frac{d}{dx} x^1 = \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x} = 1$

$$= 1 \cdot x^{1-1}$$

S $\frac{d}{dx} x^n = n x^{n-1}$ holds for n

Show $\frac{d}{dx} x^{n+1} = (n+1) x^{(n+1)-1}$

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx}(x^n \cdot x) = \left(\frac{d}{dx} x^n\right) \cdot x + x^n \cdot \frac{d}{dx} x$$

$$= n x^{n-1} \cdot x + x^n \cdot 1 = n x^n + x^n$$

$$= (n+1) x^n$$

$\therefore$ $\frac{d}{dx} x^n = n x^{n-1}$ holds for all $n \in \mathbb{Z}^+$

Alternate Proof with binomial theorem

$$(x+\Delta x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \Delta x^{n-k}$$

$$= \Delta x^n + n x \Delta x^{n-1} + n(n-1) x^2 \Delta x^{n-2} + \dots + n x^{n-1} \Delta x + x^n$$

$$\frac{d}{dx} x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^n + n x \Delta x^{n-1} + \dots + n x^{n-1} \Delta x + x^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x^{n-1} + n x \Delta x^{n-2} + \dots + n(n-1) x^{n-2} \Delta x + n x^{n-1}$$

$$= n x^{n-1}$$

7) Let x be even $\Rightarrow \exists\, k \in \mathbb{Z} \ni x = 2k$

$x^2 = (2k)^2 = 2(2k^2)$ since $k \in \mathbb{Z} \Rightarrow 2k^2 \in \mathbb{Z}$

$\Rightarrow \exists\, n = 2k^2 \in \mathbb{Z} \ni x^2 = 2n$

$\therefore$ x^2 is even

8) $\Rightarrow$ Let x be even $\Rightarrow \exists\, k \in \mathbb{Z} \ni x = 2k$

$\Rightarrow x+1 = 2k+1 \Rightarrow x+1$ is odd

$\Leftarrow$ Let $x+1$ be odd $\Rightarrow \exists\, k \in \mathbb{Z} \ni x+1 = 2k+1$

$\Rightarrow x = 2k \Rightarrow x$ is even since $k \in \mathbb{Z}$.

9) Let $x \in \mathbb{Z}$

$x^2 + x = x(x+1)$

① Let x be odd $\Rightarrow x+1$ is even

$\Rightarrow \exists\, k \in \mathbb{Z} \ni x+1 = 2k$

$\Rightarrow x^2+x = x(2k) = 2(xk)$

Since $x, k \in \mathbb{Z} \Rightarrow xk \in \mathbb{Z} \Rightarrow x^2+x$ is even.

② Let x be even $\Rightarrow \exists\, k \in \mathbb{Z} \ni x = 2k$

$\Rightarrow x^2+x = 2k(x+1) = 2(k(x+1))$

Since $x, k \in \mathbb{Z} \Rightarrow k(x+1) \in \mathbb{Z} \Rightarrow x^2+x$ is even

10) Let x be a rational # and y be an irrational #

Suppose $x+y$ is rational

then $(x+y) - x$ is rational (since the rational #s are closed under addition)

$\Rightarrow x+y-x = y$ is a rational # $\Rightarrow \Leftarrow$

$\therefore x+y$ is not rational

11) $n=1$ $\prod_{i=1}^{1} a_i = a_1 = 0 \Rightarrow \exists\, k \in \{1\}$, where $k=1$ such that $a_k = a_1 = 0$

Suppose true for n $\therefore$ if $\prod_{i=1}^{n} a_i = 0 \Rightarrow \exists\, k \in \{1, \dots, n\}$ where $a_k = 0$

Consider if $\prod_{i=1}^{n+1} a_i = 0$ then $a_1 a_2 \cdots a_n a_{n+1} = 0$

So either $a_1 a_2 \cdots a_n = 0$ or $a_{n+1} = 0$

if $a_{n+1} = 0$ then $\exists\, k \in \{1, \dots, n+1\} \ni a_k = 0$

if $a_{n+1} \neq 0$ then $a_1 a_2 \dots a_n = 0$

and by the inductive hypothesis

$\exists\, k \in \{1, 2, \ldots, n\} \ni a_k = 0$

$\therefore \exists\, k \in \{1, \ldots, n+1\} \ni a_k = 0$

and the result holds.

12) ① Let $x \leq 0$ then any $y \neq 0$

$\Rightarrow y^2 > x$ since $y^2 \geq 0 \ \forall y \in \mathbb{R}$

② Let $x > 0$

consider $y = \sqrt{x} + 1$

then $y^2 = (\sqrt{x}+1)^2 = x + 2\sqrt{x} + 1$

Since $x > 0 \Rightarrow 2\sqrt{x} + 1 > 0$

$\Rightarrow y^2 = x + (2\sqrt{x}+1) > x + 0 = x$