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Since we relied on the product and chain rules to prove the power rule, let's consider their proofs. Note that we can show there is no circular reasoning in our proofs of the power rule by showing that we don't need the power rule to prove the product or chain rules.

**Definition.** The derivative of  $f$  at  $x$  is given by  $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$  provided the limit exists.

**Theorem 1.** The derivative of  $f$  at  $x$  is given by  $f'(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$  provided the limit exists.

Proof Type: \_\_\_\_\_

*Proof.* Given  $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ , where the limit

exists, consider  $c = x + \Delta x$ .

If  $\Delta x \rightarrow 0$ , then  $x \rightarrow c$  must hold. Then

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} \\ &= \lim_{x \rightarrow c} \frac{f(c) - f(x)}{c - x} \\ &= \lim_{x \rightarrow c} \frac{-(f(x) - f(c))}{-(x - c)} \\ &= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \end{aligned}$$

□

**Theorem 2.** Let  $f$  and  $g$  be differentiable function. Then  $\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$ .

Proof Type: \_\_\_\_\_

*Proof.*  $\frac{d}{dx} [f(x)g(x)]$

$$\begin{aligned}
 &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x) + f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left( f(x + \Delta x) \frac{g(x + \Delta x) - g(x)}{\Delta x} \right. \\
 &\quad \left. + g(x) \frac{f(x + \Delta x) - f(x)}{\Delta x} \right) \\
 &= \lim_{\Delta x \rightarrow 0} f(x + \Delta x) \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
 &\quad + \lim_{\Delta x \rightarrow 0} g(x) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= f(x)g'(x) + f'(x)g(x)
 \end{aligned}$$

□

**Theorem 3.** If  $y = f(u)$  is a differentiable function of  $u$  and  $u = g(x)$  is a differentiable function of  $x$ , then  $y = f(g(x))$  is a differentiable function of  $x$  and

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \text{ or } \frac{d}{dx} (f(g(x))) = f'(g(x))g'(x).$$

Proof Type: \_\_\_\_\_

*Proof.* Let  $h(x) = f(g(x))$  and consider

$$\begin{aligned}
 h'(x) &= \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c} \\
 h'(c) &= \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{x - c} \\
 &= \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \text{ where } g(x) \neq g(c) \\
 &= \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \\
 &= f'(g(c))g'(c)
 \end{aligned}$$

□

**Definition.**  $A - B = \{x | x \in A \text{ and } x \notin B\}$

**Theorem 4.** *De Morgan's Laws:*  $A - (B \cup C) = (A - B) \cap (A - C)$

Proof Type: \_\_\_\_\_

*Proof.* Let  $x \in A - (B \cup C)$ .

$$x \in A - (B \cup C) \iff x \in A \text{ and } x \notin B \cup C$$

$$\iff x \in A \text{ and } x \notin B \text{ and } x \notin C$$

$$\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C)$$

$$\iff x \in A - B \text{ and } x \in A - C$$

$$\iff x \in (A - B) \cap (A - C)$$

Therefore  $A - (B \cup C) = (A - B) \cap (A - C)$  □

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**Theorem 5.** For  $n \in \mathbb{Z}^{\geq}$ ,  $2^{3n+1} + 5$  is always a multiple of 7.

Proof Type: \_\_\_\_\_

*Proof.* Consider  $n = 0$ ,  $2^{3(0)+1} + 5 = 7 = 7(1)$ . Since  $1 \in \mathbb{Z}$ ,  $2^{3n+1} + 5$

is a multiple of 7 for  $n = 0$ .

Suppose  $\exists m \in \mathbb{Z}$  such that  $2^{3k+1} + 5 = 7m$ .

Then  $2^{3k+1} = 7m - 5$ , where  $m \in \mathbb{Z}$ .

Consider  $2^{3(k+1)+1} + 5$ :

$$\begin{aligned} 2^{3(k+1)+1} + 5 &= 2^{3k+1+3} + 5 = 2^{3k+1} \cdot 2^3 + 5 \\ &= 8(7m - 5) + 5 = 7(8m) - 35 \\ &= 7(8m - 5) \end{aligned}$$

Since  $m \in \mathbb{Z}$ , then  $8m - 5 \in \mathbb{Z}$  and  $\exists m' = 8m - 5 \in \mathbb{Z}$

such that  $2^{3(k+1)+1} + 5 = 7m'$ .

Therefore for any  $n \in \mathbb{Z}^{\geq}$ , there exists  $m \in \mathbb{Z}$  such that

$2^{3k+1} + 5 = 7m$ , and  $2^{3k+1} + 5$  is a multiple of 7.  $\square$

**Theorem 6.**  $\sqrt{2} + \sqrt{6} < \sqrt{15}$

Proof Type: \_\_\_\_\_

*Proof.* Suppose  $\sqrt{2} + \sqrt{6} \geq \sqrt{15}$ . Then

$$\begin{aligned} & (\sqrt{2} + \sqrt{6})^2 \geq \sqrt{15}^2 \\ \implies & 8 + 2\sqrt{12} \geq 15 \\ \implies & \sqrt{12} \geq 3.5 \\ \implies & 12 \geq 3.5^2 = 12.25 \quad \implies \Leftarrow \end{aligned}$$

Therefore  $\sqrt{2} + \sqrt{6} < \sqrt{15}$ . □

**Theorem 7.** For all  $x \in \mathbb{Z}$ ,  $x(x+1)$  is even.

Proof Type: \_\_\_\_\_

*Proof.* Let  $x \in \mathbb{Z}$ . Consider  $x$  even. Then  $\exists k \in \mathbb{Z}$  such that  $x = 2k$ , and  $x(x+1) = 2k(2k+1)$ .

Since  $k \in \mathbb{Z}$ ,  $k(2k+1) \in \mathbb{Z}$  and  $x(x+1)$  is even.

Consider  $x$  odd. Then  $\exists k \in \mathbb{Z}$  such that  $x = 2k+1$ , and  $x(x+1) = (2k+1)(2k+2) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$ .

Since  $k \in \mathbb{Z}$ ,  $(2k^2 + 3k + 1) \in \mathbb{Z}$  and  $x(x+1)$  is even. □

**Theorem 8.** The smallest element of a nonempty set of positive integers is unique.

Proof Type: \_\_\_\_\_

*Proof.* Let  $S$  be a nonempty set of positive integers. Let  $a, b \in S$  such that  $a$  and  $b$  are two distinct smallest elements of  $S$ .

$$\begin{aligned} \implies & a \leq x \text{ for all } x \in S \text{ and } b \leq x \text{ for all } x \in S \\ \implies & a \leq b \text{ and } b \leq a \\ \implies & a = b \text{ and the smallest element of } S \text{ is unique} \quad \square \end{aligned}$$

**Definition.** A homomorphism  $\phi$  from a group  $G$  to a group  $\bar{G}$  is a mapping from  $G$  into  $\bar{G}$  that preserves the group operation,  $\circ$ ; that is,  $\phi(a \circ b) = \phi(a) \circ \phi(b)$  for all  $a, b \in G$ .

**Theorem 9.** Let  $G$  be the group of all polynomials with real coefficients under addition. For each  $f \in G$ , let  $\int f$  denote the antiderivative of  $f$  that passes through the point  $(0, 0)$ . Then  $\phi : f \rightarrow \int f$  is a homomorphism.

Proof Type: \_\_\_\_\_

*Proof.* Let  $f, g \in G$ , then

$\exists a_i \in \mathbb{R}$  for  $i = 1, \dots, n$  such that

$$f = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n = \sum_{i=0}^n a_ix^i \text{ and}$$

$\exists b_i \in \mathbb{R}$  for  $i = 1, \dots, m$  such that

$$g = b_0 + b_1x + b_2x^2 + \dots + b_{m-1}x^{m-1} + b_mx^m = \sum_{i=0}^m b_ix^i.$$

WLOG let  $m \geq n$  and set  $a_i = 0$  for  $i = n + 1, \dots, m$ .

Then  $f = \sum_{i=0}^m a_ix^i$ , and we have

$$\begin{aligned} \phi(f) &= \int f = \int f dx = \int \sum_{i=0}^m a_ix^i dx \\ &= \sum_{i=0}^m \int a_ix^i dx = \sum_{i=0}^m a_i \frac{x^{i+1}}{i+1}, \end{aligned}$$

Similarly  $\phi(g) = \int g = \sum_{i=0}^m b_i \frac{x^{i+1}}{i+1}$ . Then

$$\begin{aligned} \phi(f \circ g) &= \phi(f + g) = \int f + g dx = \int \sum_{i=0}^m (a_i + b_i)x^i dx \\ &= \sum_{i=0}^m \int (a_i + b_i)x^i dx = \sum_{i=0}^m (a_i + b_i) \frac{x^{i+1}}{i+1} \\ &= \sum_{i=0}^m \left( a_i \frac{x^{i+1}}{i+1} + b_i \frac{x^{i+1}}{i+1} \right) = \sum_{i=0}^m a_i \frac{x^{i+1}}{i+1} + \sum_{i=0}^m b_i \frac{x^{i+1}}{i+1} \\ &= \phi(f) + \phi(g) = \phi(f) \circ \phi(g) \end{aligned}$$

□

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**Theorem 10.** *For any rational number  $a$  and irrational number  $b$ ,  $a + b$  is irrational, and if  $a \neq 0$ , then  $ab$  is also irrational.*

Proof Type: \_\_\_\_\_

*Proof.* Let  $a$  be rational and  $b$  be irrational.

Since  $a$  is rational  $\exists m, n \in \mathbb{Z}$  such that  $a = \frac{m}{n}$ .

Suppose  $a + b$  is rational. Then  $\exists p, q \in \mathbb{Z}$  such that  $a + b = \frac{p}{q}$ .

$$\begin{aligned} \text{Then} \quad &\implies \frac{m}{n} + b = \frac{p}{q} \\ &\implies b = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} \end{aligned}$$

Since  $m, n, p, q \in \mathbb{Z}$  then  $pn - mq, qn \in \mathbb{Z}$ .

Therefore  $b$  is rational.  $\implies \Leftarrow$

Let  $a \neq 0$ , and suppose  $ab$  is rational. (Complete the proof yourself.)

Therefore \_\_\_\_\_

□

**Theorem 11.** *There exists some  $k \in \mathbb{Z}$  such that for all  $n > k$ ,  $2^n < n!$ .*

Proof Type: \_\_\_\_\_

*Proof.* Consider  $k = 3$  and  $n = 4$ , then  $2^4 = 16 < 24 = 4!$ .

Suppose for some  $m \in \mathbb{Z}$  where  $m \geq 4$ ,  $2^m < m!$  holds.

$$\begin{aligned} \text{Consider} \quad 2^{m+1} &= 2^m \cdot 2 \\ &< m! \cdot 2 \\ &< m!(m+1) \\ &= (m+1)! \end{aligned}$$

Therefore for all  $n \in \mathbb{Z}$  such that  $n \geq 4$ ,  $2^n < n!$  holds. □

**Theorem 12.** *Let  $m, n \in \mathbb{Z}$ . If  $mn$  is odd, then  $m$  and  $n$  are odd.*

Proof Type: \_\_\_\_\_

*Proof.* Let  $m, n \in \mathbb{Z}$ .

Suppose that at least one of  $m$  or  $n$  is even.

WLOG, let  $m$  be even.

Then there exists  $k \in \mathbb{Z}$  such that  $m = 2k$ .

Then  $mn = 2kn$ . Since  $k, n \in \mathbb{Z}$ , then  $kn \in \mathbb{Z}$  and  $mn$  is even.

Therefore if  $mn$  is not even, then both  $m$  and  $n$  are not even. □

**Definition.** *Given sets  $A$  and  $B$ ,  $A - B = \{x | x \in A \text{ and } x \notin B\}$  and  $A \cap B = \{x | x \in A \text{ and } x \in B\}$ .*

**Theorem 13.**  $A \cap (B - C) = (A \cap B) - (A \cap C)$

*Proof.* Let  $x \in A \cap (B - C)$ .

Proof Type: \_\_\_\_\_

$$\begin{aligned} x \in A \cap (B - C) &\iff x \in A \text{ and } x \in B - C \\ &\implies x \in A \text{ and } x \in B \text{ and } x \notin C \\ &\implies (x \in A \text{ and } x \in B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\implies x \in A \cap B \text{ and } x \notin A \cap C \\ &\implies x \in (A \cap B) - (A \cap C) \end{aligned}$$

Therefore \_\_\_\_\_

□

Is this proof complete? Why or why not?

**Definition.** Given a set  $A$ , then  $A' = \{x|x \notin A\}$ .

Given sets  $A$  and  $B$ ,  $A - B = \{x|x \in A \text{ and } x \notin B\}$ ,  $A \cap B = \{x|x \in A \text{ and } x \in B\}$ , and  $A \cup B = \{x|x \in A \text{ or } x \in B\}$ .

**Theorem 14.** Prove or disprove  $A \cup (B - C) = (A \cup B) - (A \cup C)$

Let  $x \in A \cup (B - C)$ .

$$\begin{aligned} x \in A \cup (B - C) &\iff x \in A \text{ or } x \in B - C \\ &\iff x \in A \text{ or } (x \in B \text{ and } x \notin C) \\ &\iff (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \notin C) \\ &\iff x \in A \cup B \text{ and } x \in A \cup C' \\ &\iff x \in (A \cup B) \cap (A \cup C') \end{aligned}$$

What does this prove or disprove?

Proof Type: \_\_\_\_\_

*Proof.* Let  $A = \{1, 2, 3, 4\}$ ,  $B = \{2, 3, 4, 5\}$ , and  $C = \{4, 5, 6, 7\}$ .

Then  $B - C = \{2, 3\}$ ,  $A \cup (B - C) = \{1, 2, 3, 4\}$ .

Also  $(A \cup B) = \{1, 2, 3, 4, 5\}$ ,  $(A \cup C) = \{1, 2, 3, 4, 5, 6, 7\}$ ,

so  $(A \cup B) - (A \cup C) = \emptyset$ .

Therefore \_\_\_\_\_.

□