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Since we relied on the product and chain rules to prove the power rule, let's consider their proofs. Note that we can show there is no circular reasoning in our proofs of the power rule by showing that we don't need the power rule to prove the product or chain rules.

Definition. *The derivative of f at x is given by* $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ *provided the limit exists.*

Theorem 1. *The derivative of f at x is given by* $f'(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ *provided the limit exists.*

Proof Type: _____________

Proof. Given $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, where the limit exists, consider $c = x + \Delta x$.

If $\Delta x \to 0$, then $x \to c$ must hold. Then

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} \\
&= \lim_{x \to c} \frac{f(c) - f(x)}{c - x} \\
&= \lim_{x \to c} \frac{-(f(x) - f(c))}{-(x - c)} \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\end{aligned}
$$

□

Theorem 2. *Let f and g be differentiable function. Then $\frac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\,g'(x) + f'(x)\,g(x)$.*

Proof Type: ____________

Proof. $\frac{d}{dx}\left[f(x)\,g(x)\right]$

$$\begin{aligned}
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)\,g(x+\Delta x) - f(x+\Delta x)\,g(x) + f(x+\Delta x)\,g(x) - f(x)\,g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\Bigg(f(x+\Delta x)\,\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&\qquad\qquad +g(x)\,\frac{f(x+\Delta x)-f(x)}{\Delta x}\Bigg)\\
&= \lim_{\Delta x\to 0} f(x+\Delta x)\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&\qquad + \lim_{\Delta x\to 0} g(x)\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= f(x)\,g'(x) + f'(x)\,g(x)
\end{aligned}$$

□

Theorem 3. *If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \text{ or } \frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\,g'(x)\,.$$

Proof Type: ____________

Proof. Let $h(x) = f(g(x))$ and consider

$$h'(x) = \lim_{x\to c}\frac{h(x)-h(c)}{x-c}.$$

$$\begin{aligned}
h'(c) &= \lim_{x\to c}\frac{f(g(x)) - f(g(c))}{x-c}\\
&= \lim_{x\to c}\frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\cdot\frac{g(x)-g(c)}{x-c} \text{ where } g(x)\neq g(c)\\
&= \lim_{x\to c}\frac{f(g(x)) - f(g(c))}{g(x)-g(c)}\cdot\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\\
&= f'(g(c))\,g'(c)
\end{aligned}$$

□

Definition. $A - B = \{x | x \in A \textit{ and } x \notin B\}$

Theorem 4. *De Morgan's Laws:* $A - (B \cup C) = (A - B) \cap (A - C)$

Proof Type: ______________

Proof. Let $x \in A - (B \cup C)$.

$$\begin{aligned}
x \in A - (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\
&\iff x \in A \text{ and } x \notin B \text{ and } x \notin C \\
&\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\
&\iff x \in A - B \text{ and } x \in A - C \\
&\iff x \in (A - B) \cap (A - C)
\end{aligned}$$

Therefore $A - (B \cup C) = (A - B) \cap (A - C)$ $\square$

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Theorem 5. *For $n \in \mathbb{Z}^{\geq}$, $2^{3n+1} + 5$ is always a multiple of 7.*

Proof Type: ____________

Proof. Consider $n = 0$, $2^{3(0)+1} + 5 = 7 = 7(1)$. Since $1 \in \mathbb{Z}$, $2^{3n+1} + 5$

is a multiple of 7 for $n = 0$.

Suppose $\exists\ m \in \mathbb{Z}$ such that $2^{3k+1} + 5 = 7m$.

Then $2^{3k+1} = 7m - 5$, where $m \in \mathbb{Z}$.

Consider $2^{3(k+1)+1} + 5$:

$$\begin{aligned} 2^{3(k+1)+1} + 5 &= 2^{3k+1+3} + 5 = 2^{3k+1} \cdot 2^3 + 5 \\ &= 8(7m - 5) + 5 = 7(8m) - 35 \\ &= 7(8m - 5) \end{aligned}$$

Since $m \in \mathbb{Z}$, then $8m - 5 \in \mathbb{Z}$ and $\exists\ m' = 8m - 5 \in \mathbb{Z}$

such that $2^{3(k+1)+1} + 5 = 7m'$.

Therefore for any $n \in \mathbb{Z}^{\geq}$, there exists $m \in \mathbb{Z}$ such that

$2^{3k+1} + 5 = 7m$, and $2^{3k+1} + 5$ is a multiple of 7. $\square$

Theorem 6. $\sqrt{2} + \sqrt{6} < \sqrt{15}$

Proof Type: ______________

Proof. Suppose $\sqrt{2} + \sqrt{6} \geq \sqrt{15}$. Then

$$(\sqrt{2} + \sqrt{6})^2 \geq \sqrt{15}^2$$

$$\implies \quad 8 + 2\sqrt{12} \geq 15$$

$$\implies \quad \sqrt{12} \geq 3.5$$

$$\implies \quad 12 \geq 3.5^2 = 12.25 \quad \Rightarrow\Leftarrow$$

Therefore $\sqrt{2} + \sqrt{6} < \sqrt{15}$. □

Theorem 7. *For all $x \in \mathbb{Z}$, $x(x+1)$ is even.*

Proof Type: ______________

Proof. Let $x \in \mathbb{Z}$. Consider x even. Then $\exists\ k \in \mathbb{Z}$ such that $x = 2k$, and $x(x+1) = 2k(2k+1)$.

Since $k \in \mathbb{Z}$, $k(2k+1) \in \mathbb{Z}$ and $x(x+1)$ is even.

Consider x odd. Then $\exists\ k \in \mathbb{Z}$ such that $x = 2k+1$, and $x(x+1) = (2k+1)(2k+2) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$.

Since $k \in \mathbb{Z}$, $(2k^2 + 3k + 1) \in \mathbb{Z}$ and $x(x+1)$ is even. □

Theorem 8. *The smallest element of a nonempty set of positive integers is unique.*

Proof Type: ______________

Proof. Let S be a nonempty set of positive integers. Let $a, b \in S$ such that a and b are two distinct smallest elements of S.

$\implies a \leq x$ for all $x \in S$ and $b \leq x$ for all $x \in S$

$\implies a \leq b$ and $b \leq a$

$\implies a = b$ and the smallest element of S is unique □

Definition. *A homomorphism ϕ from a group G to a group $\bar{G}$ is a mapping from G into $\bar{G}$ that preserves the group operation, $\circ$; that is, $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for all $a, b \in G$.*

Theorem 9. *Let G be the group of all polynomials with real coefficients under addition. For each $f \in G$, let $\int f$ denote the antiderivative of f that passes through the point (0, 0). Then $\phi : f \to \int f$ is a homomorphism.*

Proof Type: ____________

Proof. Let $f, g \in G$, then

$\exists$ $a_i \in \mathbb{R}$ for $i = 1, \ldots, n$ such that

$f = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1}x^{n-1} + a_n x^n = \sum_{i=0}^{n} a_i x^i$ and

$\exists$ $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$ such that

$g = b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-1}x^{m-1} + b_m x^m = \sum_{i=0}^{m} b_i x^i$.

WLOG let $m \geq n$ and set $a_i = 0$ for $i = n+1, \ldots, m$.

Then $f = \sum_{i=0}^{m} a_i x^i$, and we have

$$\begin{aligned}
\phi(f) &= \int f = \int f dx = \int \sum_{i=0}^{m} a_i x^i dx \\
&= \sum_{i=0}^{m} \int a_i x^i dx = \sum_{i=0}^{m} a_i \frac{x^{i+1}}{i+1},
\end{aligned}$$

Similarly $\phi(g) = \int g = \sum_{i=0}^{m} b_i \frac{x^{i+1}}{i+1}$. Then

$$\begin{aligned}
\phi(f \circ g) &= \phi(f+g) = \int f + g dx = \int \sum_{i=0}^{m} (a_i + b_i) x^i dx \\
&= \sum_{i=0}^{m} \int (a_i + b_i) x^i dx = \sum_{i=0}^{m} (a_i + b_i) \frac{x^{i+1}}{i+1} \\
&= \sum_{i=0}^{m} \left(a_i \frac{x^{i+1}}{i+1} + b_i \frac{x^{i+1}}{i+1} \right) = \sum_{i=0}^{m} a_i \frac{x^{i+1}}{i+1} + \sum_{i=0}^{m} b_i \frac{x^{i+1}}{i+1} \\
&= \phi(f) + \phi(g) = \phi(f) \circ \phi(g)
\end{aligned}$$

$\square$

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Theorem 10. *For any rational number a and irrational number b, $a+b$ is irrational, and if $a \neq 0$, then ab is also irrational.*

Proof Type: ______________

Proof. Let a be rational and b be irrational.

Since a is rational $\exists\ m, n \in \mathbb{Z}$ such that $a = \frac{m}{n}$.

Suppose $a+b$ is rational. Then $\exists\ p, q \in \mathbb{Z}$ such that $a+b = \frac{p}{q}$.

$$\text{Then} \quad \implies \quad \frac{m}{n} + b = \frac{p}{q}$$

$$\implies \quad b = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn}$$

Since $m, n, p, q \in \mathbb{Z}$ then $pn - mq, qn \in \mathbb{Z}$.

Therefore b is rational. $\implies\!\Longleftarrow$

Let $a \neq 0$, and suppose ab is rational. (Complete the proof yourself.)

Therefore ____ $\square$

Theorem 11. *There exists some $k \in \mathbb{Z}$ such that for all $n > k$, $2^n < n!$.*

Proof Type: ____

Proof. Consider $k = 3$ and $n = 4$, then $2^4 = 16 < 24 = 4!$.

Suppose for some $m \in \mathbb{Z}$ where $m \geq 4$, $2^m < m!$ holds.

Consider
$$\begin{aligned} 2^{m+1} &= 2^m \cdot 2 \\ &< m! \cdot 2 \\ &< m!(m+1) \\ &= (m+1)! \end{aligned}$$

Therefore for all $n \in \mathbb{Z}$ such that $n \geq 4$, $2^n < n!$ holds. □

Theorem 12. *Let $m, n \in \mathbb{Z}$. If mn is odd, then m and n are odd.*

Proof Type: ____

Proof. Let $m, n \in \mathbb{Z}$.

Suppose that at least one of m or n is even.

WLOG, let m be even.

Then there exists $k \in \mathbb{Z}$ such that $m = 2k$.

Then $mn = 2kn$. Since $k, n \in \mathbb{Z}$, then $kn \in \mathbb{Z}$ and mn is even.

Therefore if mn is not even, then both m and n are not even. □

Definition. *Given sets A and B, $A - B = \{x | x \in A \text{ and } x \notin B\}$ and $A \cap B = \{x | x \in A \text{ and } x \in B\}$.*

Theorem 13. $A \cap (B - C) = (A \cap B) - (A \cap C)$

Proof. Let $x \in A \cap (B - C)$.

Proof Type: ____

$$\begin{aligned} x \in A \cap (B - C) &\iff x \in A \text{ and } x \in B - C \\ &\implies x \in A \text{ and } x \in B \text{ and } x \notin C \\ &\implies (x \in A \text{ and } x \in B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\implies x \in A \cap B \text{ and } x \notin A \cap C \\ &\implies x \in (A \cap B) - (A \cap C) \end{aligned}$$

Therefore ____ ___ ____

□

Is this proof complete? Why or why not?

Definition. *Given a set A, then $A' = \{x | x \notin A\}$.*

Given sets A and B, $A - B = \{x | x \in A$ and $x \notin B\}$, $A \cap B = \{x | x \in A$ and $x \in B\}$, and $A \cup B = \{x | x \in A$ or $x \in B\}$.

Theorem 14. *Prove or disprove $A \cup (B - C) = (A \cup B) - (A \cup C)$*

Let $x \in A \cup (B - C)$.

$$\begin{aligned} x \in A \cup (B - C) &\iff x \in A \text{ or } x \in B - C \\ &\iff x \in A \text{ or } (x \in B \text{ and } x \notin C) \\ &\iff (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \notin C) \\ &\iff x \in A \cup B \text{ and } x \in A \cup C' \\ &\iff x \in (A \cup B) \cap (A \cup C') \end{aligned}$$

What does this prove or disprove?

Proof Type: ______________

Proof. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$, and $C = \{4, 5, 6, 7\}$.

Then $B - C = \{2, 3\}$, $A \cup (B - C) = \{1, 2, 3, 4\}$.

Also $(A \cup B) = \{1, 2, 3, 4, 5\}$, $(A \cup C) = \{1, 2, 3, 4, 5, 6, 7\}$,

so $(A \cup B) - (A \cup C) = \emptyset$.

Therefore ____________________________. □