A Proofs of Selected Theorems

THEOREM 1.2 Properties of Limits (Properties 2, 3, 4, and 5) (page 59)

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K.$$

2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$

5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof To prove Property 2, choose $\varepsilon > 0$. Because $\varepsilon/2 > 0$, you know that there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \varepsilon/2$. You also know that there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $|g(x) - K| < \varepsilon/2$. Let δ be the smaller of δ_1 and δ_2; then $0 < |x - c| < \delta$ implies that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - K| < \frac{\varepsilon}{2}.$$

So, you can apply the triangle inequality to conclude that

$$|[f(x) + g(x)] - (L + K)| \leq |f(x) - L| + |g(x) - K| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies that

$$\lim_{x\to c} [f(x) + g(x)] = L + K = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

The proof that

$$\lim_{x\to c} [f(x) - g(x)] = L - K$$

is similar.

To prove Property 3, given that

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

you can write

$$f(x)g(x) = [f(x) - L][g(x) - K] + [Lg(x) + Kf(x)] - LK.$$

Because the limit of $f(x)$ is L, and the limit of $g(x)$ is K, you have

$$\lim_{x\to c} [f(x) - L] = 0 \quad \text{and} \quad \lim_{x\to c} [g(x) - K] = 0.$$

Let $0 < \varepsilon < 1$. Then there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$|f(x) - L - 0| < \varepsilon \quad \text{and} \quad |g(x) - K - 0| < \varepsilon$$

which implies that

$$|[f(x) - L][g(x) - K] - 0| = |f(x) - L|\,|g(x) - K| < \varepsilon\varepsilon < \varepsilon.$$

So,

$$\lim_{x \to c} [f(x) - L][g(x) - K] = 0.$$

Furthermore, by Property 1, you have

$$\lim_{x \to c} Lg(x) = LK \quad \text{and} \quad \lim_{x \to c} Kf(x) = KL.$$

Finally, by Property 2, you obtain

$$\begin{aligned}\lim_{x \to c} f(x)g(x) &= \lim_{x \to c} [f(x) - L][g(x) - K] + \lim_{x \to c} Lg(x) + \lim_{x \to c} Kf(x) - \lim_{x \to c} LK \\ &= 0 + LK + KL - LK \\ &= LK.\end{aligned}$$

To prove Property 4, note that it is sufficient to prove that

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{K}.$$

Then you can use Property 3 to write

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} f(x) \frac{1}{g(x)} = \lim_{x \to c} f(x) \cdot \lim_{x \to c} \frac{1}{g(x)} = \frac{L}{K}.$$

Let $\varepsilon > 0$. Because $\lim_{x \to c} g(x) = K$, there exists $\delta_1 > 0$ such that if

$$0 < |x - c| < \delta_1, \text{ then } |g(x) - K| < \frac{|K|}{2}$$

which implies that

$$|K| = |g(x) + [|K| - g(x)]| \le |g(x)| + ||K| - g(x)| < |g(x)| + \frac{|K|}{2}.$$

That is, for $0 < |x - c| < \delta_1$,

$$\frac{|K|}{2} < |g(x)| \quad \text{or} \quad \frac{1}{|g(x)|} < \frac{2}{|K|}.$$

Similarly, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then

$$|g(x) - K| < \frac{|K|^2}{2}\varepsilon.$$

Let δ be the smaller of δ_1 and δ_2. For $0 < |x - c| < \delta$, you have

$$\left|\frac{1}{g(x)} - \frac{1}{K}\right| = \left|\frac{K - g(x)}{g(x)K}\right| = \frac{1}{|K|} \cdot \frac{1}{|g(x)|}\,|K - g(x)| < \frac{1}{|K|} \cdot \frac{2}{|K|}\,\frac{|K|^2}{2}\varepsilon = \varepsilon.$$

So, $\displaystyle\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{K}$.

Finally, the proof of Property 5 can be obtained by a straightforward application of mathematical induction coupled with Property 3. ∎

THEOREM 1.4 The Limit of a Function Involving a Radical (page 60)

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Consider the case for which $c > 0$ and n is any positive integer. For a given $\varepsilon > 0$, you need to find $\delta > 0$ such that

$$\left|\sqrt[n]{x} - \sqrt[n]{c}\right| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

which is the same as saying

$$-\varepsilon < \sqrt[n]{x} - \sqrt[n]{c} < \varepsilon \quad \text{whenever} \quad -\delta < x - c < \delta.$$

Assume $\varepsilon < \sqrt[n]{c}$, which implies that $0 < \sqrt[n]{c} - \varepsilon < \sqrt[n]{c}$. Now, let δ be the smaller of the two numbers.

$$c - \left(\sqrt[n]{c} - \varepsilon\right)^n \quad \text{and} \quad \left(\sqrt[n]{c} + \varepsilon\right)^n - c$$

Then you have

$$\begin{aligned}
-\delta < x - c &< \delta \\
-\left[c - \left(\sqrt[n]{c} - \varepsilon\right)^n\right] < x - c &< \left(\sqrt[n]{c} + \varepsilon\right)^n - c \\
\left(\sqrt[n]{c} - \varepsilon\right)^n - c < x - c &< \left(\sqrt[n]{c} + \varepsilon\right)^n - c \\
\left(\sqrt[n]{c} - \varepsilon\right)^n < x &< \left(\sqrt[n]{c} + \varepsilon\right)^n \\
\sqrt[n]{c} - \varepsilon < \sqrt[n]{x} &< \sqrt[n]{c} + \varepsilon \\
-\varepsilon < \sqrt[n]{x} - \sqrt[n]{c} &< \varepsilon.
\end{aligned}$$

THEOREM 1.5 The Limit of a Composite Function (page 61)

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof For a given $\varepsilon > 0$, you must find $\delta > 0$ such that

$$|f(g(x)) - f(L)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Because the limit of $f(x)$ as $x \to L$ is $f(L)$, you know there exists $\delta_1 > 0$ such that

$$|f(u) - f(L)| < \varepsilon \quad \text{whenever} \quad |u - L| < \delta_1.$$

Moreover, because the limit of $g(x)$ as $x \to c$ is L, you know there exists $\delta > 0$ such that

$$|g(x) - L| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Finally, letting $u = g(x)$, you have

$$|f(g(x)) - f(L)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

THEOREM 1.7 Functions That Agree at All But One Point (page 62)

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Let L be the limit of $g(x)$ as $x \to c$. Then, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(x) = g(x)$ in the open intervals $(c - \delta, c)$ and $(c, c + \delta)$, and

$$|g(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Because $f(x) = g(x)$ for all x in the open interval other than $x = c$, it follows that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

So, the limit of $f(x)$ as $x \to c$ is also L. ■

THEOREM 1.8 The Squeeze Theorem (page 65)

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof For $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|h(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_1$$

and

$$|g(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_2.$$

Because $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, there exists $\delta_3 > 0$ such that $h(x) \leq f(x) \leq g(x)$ for $0 < |x - c| < \delta_3$. Let δ be the smallest of δ_1, δ_2, and δ_3. Then, if $0 < |x - c| < \delta$, it follows that $|h(x) - L| < \varepsilon$ and $|g(x) - L| < \varepsilon$, which implies that

$$\begin{aligned} -\varepsilon < h(x) - L < \varepsilon \quad &\text{and} \quad -\varepsilon < g(x) - L < \varepsilon \\ L - \varepsilon < h(x) \quad &\text{and} \quad g(x) < L + \varepsilon. \end{aligned}$$

Now, because $h(x) \leq f(x) \leq g(x)$, it follows that $L - \varepsilon < f(x) < L + \varepsilon$, which implies that $|f(x) - L| < \varepsilon$. Therefore,

$$\lim_{x \to c} f(x) = L.$$ ■

THEOREM 1.11 Properties of Continuity (page 75)

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}, \ g(c) \neq 0$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Because f and g are continuous at $x = c$, you can write

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c).$$

For Property 1, when b is a real number, it follows from Theorem 1.2 that

$$\lim_{x \to c} [(bf)(x)] = \lim_{x \to c} [bf(x)] = b \lim_{x \to c} [f(x)] = b\,f(c) = (bf)(c).$$

Thus, bf is continuous at $x = c$.

For Property 2, it follows from Theorem 1.2 that

$$\begin{aligned} \lim_{x \to c} (f \pm g)(x) &= \lim_{x \to c} [f(x) \pm g(x)] \\ &= \lim_{x \to c} [f(x)] \pm \lim_{x \to c} [g(x)] \\ &= f(c) \pm g(c) \\ &= (f \pm g)(c). \end{aligned}$$

Thus, $f \pm g$ is continuous at $x = c$.

For Property 3, it follows from Theorem 1.2 that

$$\begin{aligned} \lim_{x \to c} (fg)(x) &= \lim_{x \to c} [f(x)g(x)] \\ &= \lim_{x \to c} [f(x)] \lim_{x \to c} [g(x)] \\ &= f(c)g(c) \\ &= (fg)(c). \end{aligned}$$

Thus, fg is continuous at $x = c$.

For Property 4, when $g(c) \neq 0$, it follows from Theorem 1.2 that

$$\begin{aligned} \lim_{x \to c} \frac{f}{g}(x) &= \lim_{x \to c} \frac{f(x)}{g(x)} \\ &= \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \\ &= \frac{f(c)}{g(c)} \\ &= \frac{f}{g}(\mathrm{c}). \end{aligned}$$

Thus, $\dfrac{f}{g}$ is continuous at $x = c$. ∎

THEOREM 1.14 Vertical Asymptotes (page 85)

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Consider the case for which $f(c) > 0$, and there exists $b > c$ such that $c < x < b$ implies $g(x) > 0$. Then for $M > 0$, choose δ_1 such that

$$0 < x - c < \delta_1 \quad \text{implies that} \quad \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

and δ_2 such that

$$0 < x - c < \delta_2 \quad \text{implies that} \quad 0 < g(x) < \frac{f(c)}{2M}.$$

Now let δ be the smaller of δ_1 and δ_2. Then it follows that

$$0 < x - c < \delta \quad \text{implies that} \quad \frac{f(x)}{g(x)} > \frac{f(c)}{2}\left[\frac{2M}{f(c)}\right] = M.$$

So, it follows that

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \infty$$

and the line $x = c$ is a vertical asymptote of the graph of h. ■

Alternative Form of the Derivative (page 101)

The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof The derivative of f at c is given by

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

Let $x = c + \Delta x$. Then $x \to c$ as $\Delta x \to 0$. So, replacing $c + \Delta x$ by x, you have

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$ ■

THEOREM 2.10 The Chain Rule (page 130)

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof In Section 2.4, you let $h(x) = f(g(x))$ and used the alternative form of the derivative to show that $h'(c) = f'(g(c))g'(c)$, provided $g(x) \neq g(c)$ for values of x other than c. Now consider a more general proof. Begin by considering the derivative of f.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

For a fixed value of x, define a function η such that

$$\eta(\Delta x) = \begin{cases} 0, & \Delta x = 0 \\ \dfrac{\Delta y}{\Delta x} - f'(x), & \Delta x \neq 0. \end{cases}$$

Because the limit of $\eta(\Delta x)$ as $\Delta x \to 0$ does not depend on the value of $\eta(0)$, you have

$$\lim_{\Delta x \to 0} \eta(\Delta x) = \lim_{\Delta x \to 0} \left[\frac{\Delta y}{\Delta x} - f'(x)\right] = 0$$

and you can conclude that η is continuous at 0. Moreover, because $\Delta y = 0$ when $\Delta x = 0$, the equation

$$\Delta y = \Delta x \eta(\Delta x) + \Delta x f'(x)$$

is valid whether Δx is zero or not. Now, by letting $\Delta u = g(x + \Delta x) - g(x)$, you can use the continuity of g to conclude that

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} [g(x + \Delta x) - g(x)] = 0$$

which implies that

$$\lim_{\Delta x \to 0} \eta(\Delta u) = 0.$$

Finally,

$$\Delta y = \Delta u \eta(\Delta u) + \Delta u f'(u) \to \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x}\eta(\Delta u) + \frac{\Delta u}{\Delta x} f'(u), \quad \Delta x \neq 0$$

and taking the limit as $\Delta x \to 0$, you have

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}\left[\lim_{\Delta x \to 0} \eta(\Delta u)\right] + \frac{du}{dx} f'(u) \\ &= \frac{du}{dx}(0) + \frac{du}{dx} f'(u) \\ &= \frac{du}{dx} f'(u) \\ &= \frac{du}{dx} \cdot \frac{dy}{du}. \end{aligned}$$

■

Concavity Interpretation (page 187)

1. Let f be differentiable on an open interval I. If the graph of f is concave *upward* on I, then the graph of f lies *above* all of its tangent lines on I.
2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Assume that f is concave upward on $I = (a, b)$. Then, f' is increasing on (a, b). Let c be a point in the interval $I = (a, b)$. The equation of the tangent line to the graph of f at c is given by

$$g(x) = f(c) + f'(c)(x - c).$$

If x is in the open interval (c, b), then the directed distance from point $(x, f(x))$ (on the graph of f) to the point $(x, g(x))$ (on the tangent line) is given by

$$\begin{aligned} d &= f(x) - [f(c) + f'(c)(x - c)] \\ &= f(x) - f(c) - f'(c)(x - c). \end{aligned}$$

Moreover, by the Mean Value Theorem there exists a number z in (c, x) such that

$$f'(z) = \frac{f(x) - f(c)}{x - c}.$$

So, you have

$$\begin{aligned} d &= f(x) - f(c) - f'(c)(x - c) \\ &= f'(z)(x - c) - f'(c)(x - c) \\ &= [f'(z) - f'(c)](x - c). \end{aligned}$$

The second factor $(x - c)$ is positive because $c < x$. Moreover, because f' is increasing, it follows that the first factor $[f'(z) - f'(c)]$ is also positive. Therefore, $d > 0$ and you can conclude that the graph of f lies above the tangent line at x. If x is in the open interval (a, c), a similar argument can be given. This proves the first statement. The proof of the second statement is similar. ∎

THEOREM 3.7 **Test for Concavity** (page 188)

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof For Property 1, assume $f''(x) > 0$ for all x in (a, b). Then, by Theorem 3.5, f' is increasing on $[a, b]$. Thus, by the definition of concavity, the graph of f is concave upward on (a, b).

For Property 2, assume $f''(x) < 0$ for all x in (a, b). Then, by Theorem 3.5, f' is decreasing on $[a, b]$. Thus, by the definition of concavity, the graph of f is concave downward on (a, b). ∎

THEOREM 3.10 Limits at Infinity (page 196)

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0.$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Begin by proving that

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

For $\varepsilon > 0$, let $M = 1/\varepsilon$. Then, for $x > M$, you have

$$x > M = \frac{1}{\varepsilon} \quad \Rightarrow \quad \frac{1}{x} < \varepsilon \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| < \varepsilon.$$

So, by the definition of a limit at infinity, you can conclude that the limit of $1/x$ as $x \to \infty$ is 0. Now, using this result, and letting $r = m/n$, you can write the following.

$$\begin{aligned}
\lim_{x\to\infty} \frac{c}{x^r} &= \lim_{x\to\infty} \frac{c}{x^{m/n}} \\
&= c\left[\lim_{x\to\infty} \left(\frac{1}{\sqrt[n]{x}}\right)^m\right] \\
&= c\left(\lim_{x\to\infty} \sqrt[n]{\frac{1}{x}}\right)^m \\
&= c\left(\sqrt[n]{\lim_{x\to\infty} \frac{1}{x}}\right)^m \\
&= c\left(\sqrt[n]{0}\right)^m \\
&= 0
\end{aligned}$$

The proof of the second part of the theorem is similar. ■

THEOREM 4.2 Summation Formulas (page 255)

1. $\sum_{i=1}^{n} c = cn, \quad c$ is a constant
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof The proof of Property 1 is straightforward. By adding c to itself n times, you obtain a sum of cn.

To prove Property 2, write the sum in increasing and decreasing order and add corresponding terms, as follows.

$$\begin{array}{rccccccccccc}
\sum_{i=1}^{n} i = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\
& \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\sum_{i=1}^{n} i = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
2\sum_{i=1}^{n} i = & \multicolumn{9}{c}{\underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ terms}}}
\end{array}$$

So,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

To prove Property 3, use mathematical induction. First, if $n = 1$, the result is true because

$$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{1(1+1)(2+1)}{6}.$$

Now, assuming the result is true for $n = k$, you can show that it is true for $n = k + 1$, as follows.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k+1}{6}(2k^2 + k + 6k + 6) \\
&= \frac{k+1}{6}[(2k+3)(k+2)] \\
&= \frac{(k+1)(k+2)[2(k+1)+1]}{6}
\end{aligned}$$

Property 4 can be proved using a similar argument with mathematical induction. ■

THEOREM 4.8 Preservation of Inequality (page 272)

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof To prove Property 1, suppose, on the contrary, that

$$\int_a^b f(x)\,dx = I < 0.$$

Then, let $a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$ be a partition of $[a, b]$, and let

$$R = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

be a Riemann sum. Because $f(x) \ge 0$, it follows that $R \ge 0$. Now, for $\|\Delta\|$ sufficiently small, you have $|R - I| < -I/2$, which implies that

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = R < I - \frac{I}{2} < 0$$

which is not possible. From this contradiction, you can conclude that

$$0 \le \int_a^b f(x)\,dx.$$

To prove Property 2, note that $f(x) \le g(x)$ implies that $g(x) - f(x) \ge 0$. So, you can apply the result of Property 1 to conclude that

$$0 \le \int_a^b [g(x) - f(x)]\,dx$$

$$0 \le \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Properties of the Natural Logarithmic Function (page 319)

The natural logarithmic function is one-to-one.

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Recall from Section P.3 that a function f is one-to-one if for x_1 and x_2 in its domain

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

Let $f(x) = \ln x$. Then $f'(x) = \dfrac{1}{x} > 0$ for $x > 0$. So f is increasing on its entire domain $(0, \infty)$ and therefore is strictly monotonic (see Section 3.3). Choose x_1 and x_2 in the domain of f such that $x_1 \neq x_2$. Because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \quad \text{or} \quad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, $f(x) = \ln x$ is one-to-one.

To verify the limits, begin by showing that $\ln 2 \geq \frac{1}{2}$. From the Mean Value Theorem for Integrals, you can write

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx = \frac{1}{c}(2 - 1) = \frac{1}{c}$$

where c is in $[1, 2]$.

This implies that

$$\begin{aligned} 1 \leq &\ c\ \leq 2 \\ 1 \geq &\ \frac{1}{c}\ \geq \frac{1}{2} \\ 1 \geq &\ \ln 2 \geq \frac{1}{2}. \end{aligned}$$

Now, let N be any positive (large) number. Because $\ln x$ is increasing, it follows that if $x > 2^{2N}$, then

$$\ln x > \ln 2^{2N} = 2N \ln 2.$$

However, because $\ln 2 \geq \frac{1}{2}$, it follows that

$$\ln x > 2N \ln 2 \geq 2N\left(\frac{1}{2}\right) = N.$$

This verifies the second limit. To verify the first limit, let $z = 1/x$. Then, $z \to \infty$ as $x \to 0^+$, and you can write

$$\begin{aligned} \lim_{x\to 0^+} \ln x &= \lim_{x\to 0^+}\left(-\ln \frac{1}{x}\right) \\ &= \lim_{z\to\infty} (-\ln z) \\ &= -\lim_{z\to\infty} \ln z \\ &= -\infty. \end{aligned}$$

■

THEOREM 5.8 Continuity and Differentiability of Inverse Functions (page 341)

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof To prove Property 1, first show that if f is continuous on I and has an inverse function, then f is strictly monotonic on I. Suppose that f were not strictly monotonic. Then there would exist numbers x_1, x_2, x_3 in I such that $x_1 < x_2 < x_3$, but $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$. Without loss of generality, assume $f(x_1) < f(x_3) < f(x_2)$. By the Intermediate Value Theorem, there exists a number x_0 between x_1 and x_2 such that $f(x_0) = f(x_3)$. So, f is not one-to-one and cannot have an inverse function. So, f must be strictly monotonic.

Because f is continuous, the Intermediate Value Theorem implies that the set of values of f

$$\{f(x): x \in I\}$$

forms an interval J. Assume that a is an interior point of J. From the previous argument, $f^{-1}(a)$ is an interior point of I. Let $\varepsilon > 0$. There exists $0 < \varepsilon_1 < \varepsilon$ such that

$$I_1 = (f^{-1}(a) - \varepsilon_1, f^{-1}(a) + \varepsilon_1) \subseteq I.$$

Because f is strictly monotonic on I_1, the set of values $\{f(x): x \in I_1\}$ forms an interval $J_1 \subseteq J$. Let $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq J_1$. Finally, if

$$|y - a| < \delta, \text{ then } |f^{-1}(y) - f^{-1}(a)| < \varepsilon_1 < \varepsilon.$$

So, f^{-1} is continuous at a. A similar proof can be given if a is an endpoint.

To prove Property 2, let y_1 and y_2 be in the domain of f^{-1}, with $y_1 < y_2$. Then, there exist x_1 and x_2 in the domain of f such that

$$f(x_1) = y_1 < y_2 = f(x_2).$$

Because f is increasing, $f(x_1) < f(x_2)$ holds precisely when $x_1 < x_2$. Therefore,

$$f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$$

which implies that f^{-1} is increasing. (Property 3 can be proved in a similar way.)

Finally, to prove Property 4, consider the limit

$$(f^{-1})'(a) = \lim_{y \to a} \frac{f^{-1}(y) - f^{-1}(a)}{y - a}$$

where a is in the domain of f^{-1} and $f^{-1}(a) = c$. Because f is differentiable on an interval containing c, f is continuous on that interval, and so is f^{-1} at a. So, $y \to a$ implies that $x \to c$, and you have

$$\begin{aligned}(f^{-1})'(a) &= \lim_{x \to c} \frac{x - c}{f(x) - f(c)} \\ &= \lim_{x \to c} \frac{1}{\left(\dfrac{f(x) - f(c)}{x - c}\right)} \\ &= \frac{1}{\displaystyle\lim_{x \to c} \frac{f(x) - f(c)}{x - c}} \\ &= \frac{1}{f'(c)}.\end{aligned}$$

So, $(f^{-1})'(a)$ exists, and f^{-1} is differentiable at $f(c)$. ■

THEOREM 5.9 The Derivative of an Inverse Function (page 341)

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof From the proof of Theorem 5.8, letting $a = x$, you know that g is differentiable. Using the Chain Rule, differentiate both sides of the equation $x = f(g(x))$ to obtain

$$1 = f'(g(x)) \frac{d}{dx}[g(x)].$$

Because $f'(g(x)) \neq 0$, you can divide by this quantity to obtain

$$\frac{d}{dx}[g(x)] = \frac{1}{f'(g(x))}.$$ ■

THEOREM 5.10 Operations with Exponential Functions (Property 2) (page 347)

Let a and b be any real numbers.

2. $\dfrac{e^a}{e^b} = e^{a-b}$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof To prove Property 2, you can write

$$\ln\left(\frac{e^a}{e^b}\right) = \ln e^a - \ln e^b = a - b = \ln(e^{a-b})$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$\frac{e^a}{e^b} = e^{a-b}.$$

THEOREM 5.15 A Limit Involving *e* (page 360)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. Taking the natural logarithm of each side, you have

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right].$$

Because the natural logarithmic function is continuous, you can write

$$\ln y = \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty}\left\{\frac{\ln[1+(1/x)]}{1/x}\right\}.$$

Letting $x = \dfrac{1}{t}$, you have

$$\begin{aligned}\ln y &= \lim_{t\to 0^+}\frac{\ln(1+t)}{t}\\ &= \lim_{t\to 0^+}\frac{\ln(1+t)-\ln 1}{t}\\ &= \frac{d}{dx}\ln x \ \text{ at } \ x = 1\\ &= \frac{1}{x} \ \text{ at } \ x = 1\\ &= 1.\end{aligned}$$

Finally, because $\ln y = 1$, you know that $y = e$, and you can conclude that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

THEOREM 5.16 Derivatives of Inverse Trigonometric Functions (arcsin u and arccos u) (page 369)

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof

Method 1: Apply Theorem 5.9.

Let $f(x) = \sin x$ and $g(x) = \arcsin x$. Because f is differentiable on

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

you can apply Theorem 5.9.

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\arcsin x)} \\ &= \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} \\ &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

If u is a differentiable function of x, then you can use the Chain Rule to write

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}, \quad \text{where} \quad u' = \frac{du}{dx}.$$

Method 2: Use implicit differentiation.

Let $y = \arccos x$, $0 \le y \le \pi$. So, $\cos y = x$, and you can use implicit differentiation.

$$\begin{aligned} \cos y &= x \\ -\sin y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{-1}{\sin y} \\ \frac{dy}{dx} &= \frac{-1}{\sqrt{1-\cos^2 y}} \\ \frac{dy}{dx} &= \frac{-1}{\sqrt{1-x^2}} \end{aligned}$$

If u is a differentiable function of x, then you can use the Chain Rule to write

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}, \quad \text{where} \quad u' = \frac{du}{dx}.$$ ■

THEOREM 8.3 The Extended Mean Value Theorem (page 558)

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof You can assume that $g(a) \neq g(b)$, because otherwise, by Rolle's Theorem, it would follow that $g'(x) = 0$ for some x in (a, b). Now, define $h(x)$ as

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(x).$$

Then

$$\begin{aligned} h(a) &= f(a) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(a) \\ &= \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)} \end{aligned}$$

and

$$\begin{aligned} h(b) &= f(b) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(b) \\ &= \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)} \end{aligned}$$

and by Rolle's Theorem there exists a point c in (a, b) such that

$$\begin{aligned} h'(c) &= f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) \\ &= 0 \end{aligned}$$

which implies that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$. ∎

THEOREM 8.4 L'Hôpital's Rule (page 558)

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

You can use the Extended Mean Value Theorem to prove L'Hôpital's Rule. Of the several different cases of this rule, the proof of only one case is illustrated. The remaining cases where $x \to c^-$ and $x \to c$ are left for you to prove.

Proof Consider the case for which $\lim\limits_{x \to c^+} f(x) = 0$ and $\lim\limits_{x \to c^+} g(x) = 0$. Define the new functions

$$F(x) = \begin{cases} f(x), & x \neq c \\ 0, & x = c \end{cases} \quad \text{and} \quad G(x) = \begin{cases} g(x), & x \neq c \\ 0, & x = c \end{cases}.$$

For any x, $c < x < b$, F and G are differentiable on $(c, x]$ and continuous on $[c, x]$. You can apply the Extended Mean Value Theorem to conclude that there exists a number z in (c, x) such that

$$\frac{F'(z)}{G'(z)} = \frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F(x)}{G(x)} = \frac{f'(z)}{g'(z)} = \frac{f(x)}{g(x)}.$$

Finally, by letting x approach c from the right, $x \to c^+$, you have $z \to c^+$ because $c < z < x$, and

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(z)}{g'(z)} = \lim_{z \to c^+} \frac{f'(z)}{g'(z)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}.$$

THEOREM 9.15 Alternating Series Remainder (page 621)

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof The series obtained by deleting the first N terms of the given series satisfies the conditions of the Alternating Series Test and has a sum of R_N.

$$\begin{aligned} R_N = S - S_N &= \sum_{n=1}^{\infty} (-1)^{n+1} a_n - \sum_{n=1}^{N} (-1)^{n+1} a_n \\ &= (-1)^N a_{N+1} + (-1)^{N+1} a_{N+2} + (-1)^{N+2} a_{N+3} + \cdots \\ &= (-1)^N (a_{N+1} - a_{N+2} + a_{N+3} - \cdots) \\ |R_N| &= a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} - \cdots \\ &= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) - \cdots \leq a_{N+1} \end{aligned}$$

Consequently, $|S - S_N| = |R_N| \leq a_{N+1}$, which establishes the theorem.

THEOREM 9.19 Taylor's Theorem (page 642)

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof To find $R_n(x)$, fix x in I $(x \neq c)$ and write $R_n(x) = f(x) - P_n(x)$, where $P_n(x)$ is the nth Taylor polynomial for $f(x)$. Then let g be a function of t defined by

$$g(t) = f(x) - f(t) - f'(t)(x - t) - \cdot \cdot \cdot - \frac{f^{(n)}(t)}{n!}(x - t)^n - R_n(x)\frac{(x - t)^{n+1}}{(x - c)^{n+1}}.$$

The reason for defining g in this way is that differentiation with respect to t has a telescoping effect. For example, you have

$$\frac{d}{dt}[-f(t) - f'(t)(x - t)] = -f'(t) + f'(t) - f''(t)(x - t) = -f''(t)(x - t).$$

The result is that the derivative $g'(t)$ simplifies to

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + (n + 1)R_n(x)\frac{(x - t)^n}{(x - c)^{n+1}}$$

for all t between c and x. Moreover, for a fixed x,

$$g(c) = f(x) - [P_n(x) + R_n(x)] = f(x) - f(x) = 0$$

and

$$g(x) = f(x) - f(x) - 0 - \cdot \cdot \cdot - 0 = f(x) - f(x) = 0.$$

Therefore, g satisfies the conditions of Rolle's Theorem, and it follows that there is a number z between c and x such that $g'(z) = 0$. Substituting z for t in the equation for $g'(t)$ and then solving for $R_n(x)$, you obtain

$$\begin{aligned} g'(z) &= -\frac{f^{(n+1)}(z)}{n!}(x - z)^n + (n + 1)R_n(x)\frac{(x - z)^n}{(x - c)^{n+1}} \\ 0 &= -\frac{f^{(n+1)}(z)}{n!}(x - z)^n + (n + 1)R_n(x)\frac{(x - z)^n}{(x - c)^{n+1}} \\ R_n(x) &= \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}. \end{aligned}$$

Finally, because $g(c) = 0$, you have

$$\begin{aligned} 0 &= f(x) - f(c) - f'(c)(x - c) - \cdot \cdot \cdot - \frac{f^{(n)}(c)}{n!}(x - c)^n - R_n(x) \\ f(x) &= f(c) + f'(c)(x - c) + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x). \end{aligned}$$ ∎

THEOREM 9.20 Convergence of a Power Series (page 648)

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, then the radius of convergence is $R = 0$. If the series converges for all x, then the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof In order to simplify the notation, the theorem for the power series $\Sigma\, a_n x^n$ centered at $x = 0$ will be proved. The proof for a power series centered at $x = c$ follows easily. A key step in this proof uses the completeness property of the set of real numbers: If a nonempty set S of real numbers has an upper bound, then it must have a least upper bound (see page 591).

It must be shown that if a power series $\Sigma\, a_n x^n$ converges at $x = d$, $d \neq 0$, then it converges for all b satisfying $|b| < |d|$. Because $\Sigma\, a_n x^n$ converges, $\lim_{n\to\infty} a_n d^n = 0$. So, there exists an integer $N > 0$ such that $|a_n d^n| < 1$ for all $n \geq N$. Then for $n \geq N$,

$$|a_n b^n| = \left|a_n b^n \frac{d^n}{d^n}\right| = |a_n d^n|\left|\frac{b^n}{d^n}\right| < \left|\frac{b^n}{d^n}\right|.$$

So, for $|b| < |d|$, $\left|\dfrac{b}{d}\right| < 1$, which implies that

$$\sum \left|\frac{b^n}{d^n}\right| = \sum \left|\frac{b}{d}\right|^n$$

is a convergent geometric series. By the Comparison Test, the series $\Sigma\, a_n b^n$ converges.

Similarly, if the power series $\Sigma\, a_n x^n$ diverges at $x = b$, where $b \neq 0$, then it diverges for all d satisfying $|d| > |b|$. If $\Sigma\, a_n d^n$ converged, then the argument above would imply that $\Sigma\, a_n b^n$ converged as well.

Finally, to prove the theorem, suppose that neither Case 1 nor Case 3 is true. Then there exist points b and d such that $\Sigma\, a_n x^n$ converges at b and diverges at d. Let $S = \{x\colon \Sigma\, a_n x^n \text{ converges}\}$. S is nonempty because $b \in S$. If $x \in S$, then $|x| \leq |d|$, which shows that $|d|$ is an upper bound for the nonempty set S. By the completeness property, S has a least upper bound, R.

Now, if $|x| > R$, then $x \notin S$, so $\Sigma\, a_n x^n$ diverges. And if $|x| < R$, then $|x|$ is not an upper bound for S, so there exists b in S satisfying $|b| > |x|$. Because $b \in S$, $\Sigma\, a_n b^n$ converges, which implies that $\Sigma\, a_n x^n$ converges. ■

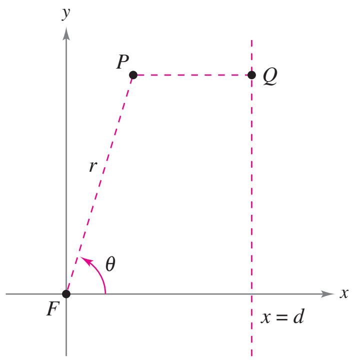
THEOREM 10.16 Classification of Conics by Eccentricity (page 734)

Let F be a fixed point (*focus*) and let D be a fixed line (*directrix*) in the plane. Let P be another point in the plane and let e (*eccentricity*) be the ratio of the distance between P and F to the distance between P and D. The collection of all points P with a given eccentricity is a conic.

1. The conic is an ellipse for $0 < e < 1$.
2. The conic is a parabola for $e = 1$.
3. The conic is a hyperbola for $e > 1$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof If $e = 1$, then, by definition, the conic must be a parabola. If $e \neq 1$, then you can consider the focus F to lie at the origin and the directrix $x = d$ to lie to the right of the origin, as shown in the figure.



For the point $P = (r, \theta) = (x, y)$, you have

$$|PF| = r \quad \text{and} \quad |PQ| = d - r\cos\theta.$$

Given that $e = \dfrac{|PF|}{|PQ|}$, it follows that

$$|PF| = |PQ|e \quad \Longrightarrow \quad r = e(d - r\cos\theta).$$

By converting to rectangular coordinates and squaring each side, you obtain

$$\begin{aligned} x^2 + y^2 &= e^2(d - x)^2 \\ &= e^2(d^2 - 2dx + x^2). \end{aligned}$$

Completing the square produces

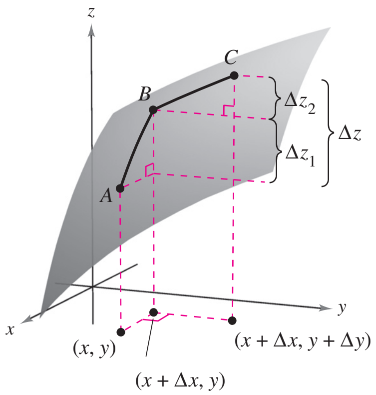
$$\left(x + \frac{e^2 d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2 d^2}{(1 - e^2)^2}.$$

If $e < 1$, then this equation represents an ellipse. If $e > 1$, then $1 - e^2 < 0$, and the equation represents a hyperbola. ∎

THEOREM 13.4 Sufficient Condition for Differentiability (page 901)

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

See LarsonCalculus.com for Bruce Edwards's video of this proof.



Proof Let S be the surface defined by $z = f(x, y)$, where f, f_x, and f_y are continuous at (x, y). Let A, B, and C be points on surface S, as shown in the figure. From this figure, you can see that the change in f from point A to point C is given by

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)]\\ &= \Delta z_1 + \Delta z_2.\end{aligned}$$

Between A and B, y is fixed and x changes. So, by the Mean Value Theorem, there is a value x_1 between x and $x + \Delta x$ such that

$$\Delta z_1 = f(x + \Delta x, y) - f(x, y) = f_x(x_1, y)\,\Delta x.$$

Similarly, between B and C, x is fixed and y changes, and there is a value y_1 between y and $y + \Delta y$ such that

$$\Delta z_2 = f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = f_y(x + \Delta x, y_1)\,\Delta y.$$

By combining these two results, you can write

$$\Delta z = \Delta z_1 + \Delta z_2 = f_x(x_1, y)\Delta x + f_y(x + \Delta x, y_1)\,\Delta y.$$

If you define ε_1 and ε_2 as $\varepsilon_1 = f_x(x_1, y) - f_x(x, y)$ and $\varepsilon_2 = f_y(x + \Delta x, y_1) - f_y(x, y)$, it follows that

$$\begin{aligned}\Delta z = \Delta z_1 + \Delta z_2 &= [\varepsilon_1 + f_x(x, y)]\,\Delta x + [\varepsilon_2 + f_y(x, y)]\,\Delta y\\ &= [f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y] + \varepsilon_1\Delta x + \varepsilon_2\Delta y.\end{aligned}$$

By the continuity of f_x and f_y and the fact that $x \le x_1 \le x + \Delta x$ and $y \le y_1 \le y + \Delta y$, it follows that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Therefore, by definition, f is differentiable.

THEOREM 13.6 Chain Rule: One Independent Variable (page 907)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof Because g and h are differentiable functions of t, you know that both Δx and Δy approach zero as Δt approaches zero. Moreover, because f is a differentiable function of x and y, you know that $\Delta w = (\partial w/\partial x)\,\Delta x + (\partial w/\partial y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$, where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. So, for $\Delta t \ne 0$

$$\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}$$

from which it follows that

$$\frac{dw}{dt} = \lim_{\Delta t \to 0}\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + 0\left(\frac{dx}{dt}\right) + 0\left(\frac{dy}{dt}\right) = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$