



The following has been prepared in conjunction with a talk given at a Spring 2019 Teach Them Diligently conference for homeschoolers. However, the driving principles apply to the learning of math in any environment and are not necessary restricted to use in a homeschooled context.

Learning the Language of the Universe: How to Study Math

Purpose: helping homeschool families better prepare their children for however God will use math in their lives

The following handout is intended to accompany the presentation given at Teach Them Diligently. The majority of the content from the slides has been included for continuity, and additional material has been included.

The following is structured around the three key ideas: goal of learning is to truly understand ideas so they can be applied in new circumstances (not acquire facts or complete tasks), this takes hard work (and our sin nature makes us naturally resistant to hard work), and there are probably better ways to accomplish our goals than what we've "always done".

We seek to minimize the math-related frustration that homeschool families may feel by providing some details regarding goals when studying math and practical tips on how to accomplish real mathematical learning.

Disclaimer: We have yet to find a way to learn well without working hard. The following suggestions will require hard work if they are to succeed, but they do actually work to learn math.

Appreciate the Value of Mathematics:

Keeping the value of a hard activity in mind makes persevering easier. So what does math really do for us? Here are just a few benefits that math brings:

1. Builds comfort and proficiency with numerical problems, so that we can handle these aspects of our lives, including items such as budgeting money, evaluating financial investments, shopping "deals" wisely (the big package isn't always cheaper), and cooking (and scaling recipes appropriately). We are better able to exercise control over the day-to-day aspects of our life, instead of being controlled by them.
2. Develops critical thinking that the Lord expects us to exercise in our daily living; teaches us to analyze a problem, to understand it and break it down into the smaller problems that can then be solved to construct an efficient solution to the large problem.
3. Teaches us to think critically about the consequences of statements. Paul would have made an excellent mathematician. He regularly constructs arguments (especially in Romans) that start with the given definitions and uses these and their logical consequences to support a foundational Christian principle. (Romans 6 is a great example, starting with definitions of a "Christian" and a "slave", Paul argues that we just can't live a lifestyle of sin if we really are a Christian. Not



embracing sin is not an arbitrary “rule” that Christians must follow, but the necessary and logical consequence of actually being redeemed.) Good biblical hermeneutics require good logical and critical thinking, and math helps with that goal.

4. Shows us a picture of our God and Creator that we wouldn't see elsewhere. Art is a reflection of the artist, and we see both practical and beautiful math all round in God's creation. By better understanding the created elements (including math), we can better understand the Creator.
 - a. Math is repeatable, consistent. God is reliable, always reacting in the same way to us.
 - b. Math requires that you act on known facts, not personal opinion or feeling. We ought to always respond to God based on what we know to be true (the facts as revealed in the scriptures), not on how we feel about the circumstance.
 - c. Math often requires us to take the first known steps trusting that the next step will become clear when we get to it. The Christian life is the same; we step forward and do what we know to be right now, and trust that God will direct the next step when we get there.
 - d. Math has ideas that are both infinite and incomprehensible while also being knowable and useful (see following example). God is this way also; He is beyond knowing in His fullness, but He has made a way for each of us to know Him and have a relationship with him.

It has been proved that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots = 2$, where the ... means you keep the pattern going forever (if you stop before forever, you won't get 2, this is called an infinite geometric sum, wonderful things, very useful in finance). But 2 is totally understanding (even if the adding up forever isn't). Our smallest children grasp 2 without even being aware that we might have to go forever to get it.

- e. Math has beauty and elegance. God is not utilitarian, valuing only that which is practical. He has created beauty throughout the world. Consider the mathematics of fractals (search on fractals or fractal images) or pi as music, both elegant and beautiful. Fractals are even practical (search on fractals among us).

No one expects math to be everyone's favorite, but as Christians we must value what God values, and evidence indicates that He has value for math.

Adjust Expectations:

Goal: to avoid frustrations by having realistic expectations

1. Remind your student regularly that:
 - Learning new ideas and developing critical thinking/problem solving skills
 - Is HARD work (and that's okay, doing hard things is rewarding)
 - Takes mental concentration and intense focus (it won't work if it only has some of your attention, so watch out for the technology, the day dreaming, the activity around you, etc. that distracts you)
 - Is a skill, and like any other skill, it is hard to do at first, usually doesn't look very pretty, but gets easier with time (and you WILL get better at it if you keep working)

- Math is a process; it IS repeatable, and if you figure out the process, you can solve problems correctly every time.
- Task completion is NOT the goal
 - Do not use “getting the homework problems done in the least amount of time” as your benchmark for success
 - Keep mastery of ideas as the goal, so as a student, adjust your “task” list
 - Review lesson/listen to lectures/etc.
 - Create study notes – ½ to 1 page explanations of how the mathematical process works in “English”
 - Work homework problems until _____ (enter reasonable goal here for your student)

Consider, completing 5 problems correctly (the first time) within _____ minutes or working 3 of each type of problem with _____ minutes or after you do 3 of a type correctly (with no help) you can move on to the next type of problem

Set a benchmark that’s based on mastery, not completion of a specified # of problems (regardless of how well they understand). You **can** connect this to a concrete number of problems **if** the student can show the following

1. They understand the process – proved by explaining it to you.
2. They can recognize when to use the process (which problems need this technique) – proved by being able to describe in general the kinds of problems for which you would use this technique in “English”.
3. They can accurately solve this kind of problem in a limited timeframe consistently – proved by short, timed quizzes.

Adjust expectations based on your students – are they missing the goal because they don’t understand how to solve it or because they are careless with numerical computation or just getting distracted? Do they need to have smaller chunks of “math” time throughout the day because their short term memory is great, but their long term is lacking?

2. Remind yourself regularly that:

- You must not remove the struggle from your student’s experience. As in all areas of our lives, we learn math primarily through struggling with it until we have understood.
 - You can help your student learn perseverance in difficulty by not removing the struggle, but by encouraging them to keep going or help them learn techniques to cope with difficult ideas.
 - You may find it helpful to consider several “math” times each day instead of doing all the math at once. Sometimes setting aside frustrating things for a little while and coming back to them helps us cope with the struggle and learn not to give up.
- You must stand firm; don’t let them wear you down into accepting anything less than actually understanding (remember the goal is understanding the process, not finding some numbers) even when they are “sure” that they know how to do the problems and understand everything without doing “all this work”.



- Their lack of experience with what the future holds will make them certain that they don't really need what you are describing (learning for long term understanding, mastering ideas [not just repeating someone else's explanation], integrating ideas from multiple places to solve more complex problems). They are sure they just need the answer.
- Persist, as they mature, they will come to appreciate your perspective and be ready for whatever God has for them.
- Remember the goal is mastery of ideas, not task completion (and it's a learned skill, so the more we practice, the better we get – it will get easier if you persist)
 - You are teaching them how new ideas are actually mastered – not an easy thing, but a powerful skill for college and the future.
 - Often students “get by” without this skill in high school typically because the pace is slower and there are often less complex problems – in college we double the pace and double the complexity (and real life is worse), so having a technique with which they can comfortably attack something new/hard/overwhelming makes the future easier to handle.
- Keep your eye on the prize: mastering ideas. Here is a general description that can help you evaluate the responses that you are hearing/seeing from your student.
 - When they simply providing a “list” of facts/information/formulas without being able to connect ideas together, they have weak understanding.
 - When they can use the facts/information/formulas to develop relationships between ideas and construct solutions to problems (sometimes with many steps), they have strong understanding.
- Each student can do this to differing levels
 - Set your goal to push them just beyond their current ability
 - Adjust your expectations to balance quantity with quality – know what absolutely must be covered in a year, what really should be covered (but we won't die if we don't), and what you can add if you run out of stuff to do at the end of the year.

Study Effectively

Goal: to learn the right stuff (not just complete tasks), so our time spent actually profits in the long run

Understand what math is and is not

... so we can actually be sure we know what success is

1. It **is a process** by which a variety of problems are solved
 - Always show your entire process of thinking (the answer is the only logical conclusion based on the work presented, not some number written on a page)
 - One line per logical step, working down a sheet of paper – it's about developing logical thinking, not finding the “answer”, but developing an iron-clad argument that your final result is correct



- If you cannot communicate the logic of your thinking, knowing it is fairly irrelevant – I tell my college students, tests are not about what you know; they are about what you can communicate about what you know
2. **It is not a numerical answer** – all the problems in textbooks are fake, answers in boxes are useless. The process is what is repeatable on problems we actually care about. Homework problems are just to ensure that we can actually implement the process correctly and efficiently. Because the process is what matters, neatness and clarity in presentation is important; therefore, problems should always be worked down the page with only one logical step per line (if you instinctively struggle with “wasting” paper, then consider that this approach is not a waste, but an investment (and paper is cheap compared to the expenses of education). If your students are still struggling with penmanship issues, encourage them to use the paper that makes it easiest to be clear (some students do better using blank white paper for math) and to ensure they put appropriate space between lines; remember clarity in expression is what we want to see (even if it takes 5 sheets of paper).

Change the Math Class Agenda

Replace the old, “traditional” agenda (read/watch lectures, work many problems, take a test) with an agenda that puts more attention on mastery and less on task completion

1. Your students must still read/watch material, but we want them to do it with new purpose.
 - The goal is to understand the main ideas (and express them in their own words) and see how the main ideas apply to an example.
 - If they’re achieving the goal above, we don’t care if they read every word in the chapter or listen to every minute of the lecture.
2. As they are reading (this can be thought of as active reading), they should create study notes that express the main ideas in their own words.
 - Don’t use complete sentences, but do have complete thoughts.
 - Make good use of bulleted lists to organize the steps of a problem.
 - Include both “English” explanations and mathematical examples.
 - Express ideas in words first, symbols/notation only come after the idea is expressed
 - Be forewarned, suave writing is not the goal here – expressing the idea in their own words is much, much more important, so don’t treat it like an English essay.
 - Be sure to include both “theory” and examples. Mathematical “theory” generally includes definitions, formulas, relationships that are the foundation to solving the problems in that section. Usually the theory can be mapped to how problems are actually solved. As Christians we learn the “theory” of how life was supposed to work based on the doctrines and principles as presented in the Bible, and we work to apply that “theory” to the “examples” in our lives. If we ignore the theory of the Bible, solving problems in life is even harder. This is also true in math; if we forgo understanding the mathematical theory that makes the problem solving work, the problems get harder to solve.
 - Identify the types of problems that will be solved using these ideas and work out an example problem.

- Write out bullet point steps that describe (in English) how to solve the problems (side by side with the example)
 - Determine how one can recognize that this particular problem gets solved with this particular technique. What characteristics will we expect the question to have (how will it look)?
 - A usual length for one topic would be $\frac{1}{2}$ to 1 page. Less than $\frac{1}{2}$ page probably means that they didn't cover everything, and more than 1 page probably means that they aren't synthesizing (just mindlessly copying stuff from somewhere without understanding why it should be there or how it relates to the other stuff).
 - Consider the quotation: "I have made this longer than usual because I have not had time to make it shorter." Blaise Pascal, philosopher and mathematician, 1657
Conciseness and precision in expressing ideas is hard and takes effort, but this is where the understanding is developed.
 - See an example: <https://math.bju.edu/media/bju-math-division/bju-math-department/melissa-gardenghi/studying-math/StudyNotesPerimeterCircumference.pdf>
3. Work problems with the following goals (examples and homework)
- To determine if they're correctly understanding the process – if they're getting the right answer on a couple of examples, then they (and their process/study notes) are probably in good shape
 - To develop speed and accuracy:
 - Speed is important since you can't afford to take 10 minutes on a 5 minute problem next year.
 - Accuracy is important because eventually the answers will matter.
4. Consider an oral quiz/teaching session
- Once your students say their homework is done, select a problem that was not assigned but in the section and have your students instruct you in how to solve it.
 - Depending on the nature of the problems, they can either
 - Verbally explain the process to you – you can judge their mastery by how clearly you can follow their explanation, how immediately the explanation comes (if they keep hesitating, send them back to their study notes and their favorite stuffed toy and have them practice teach it), and if it accurately solves the problem.
This is better for pattern recognition problems that don't actually have "work", for example,
 - Identify numbers that are rational, irrational, and integer.
 - When should we use angle-side-angle in geometry relationships?
 - Or tell you each step to take – so get out a sheet of paper, and you do exactly what they say and only what they say (don't fill in any details for them, and don't prompt them with anything more than a "what do I do next" or "is this what you wanted me to do")
This strategy works better when the problems require some sort of arithmetic or algebra.
5. Take a unit test. Testing is beneficial; it forces us to be accountable for retaining understanding over time. Remember we have the goal to assess mastery, speed, and accuracy.



- To assess mastery, ensure there are problems on your tests that are “basic” execution problems as well as at least a few problems that force the student to apply the process to a problem in a new/unfamiliar way.
- To assess sufficient speed, I strongly recommend a timed environment. Not being prepared for timed assessments are one of the challenges that many homeschooled students face in our collegiate classrooms. Your curriculum may suggest some guidelines, or you can use a rough estimate of a “traditional classroom” time-frame.
 - For early elementary students, 15 to 20 minutes for a test is probably reasonable.
 - For upper elementary students, a gradual shift from the 20 minutes to 40 minutes is probably reasonable.
 - For middle school to high school students, they probably need 40 to 50 minutes for a test.
- To assess accuracy, require your student’s grade to be based solely on the quality of the answers submitted at the end of the timed testing period.
 - This teaches your students to be prepared to work/perform in a world where there aren’t “do-overs” without seriously marring their future.
 - This improves efficiency by encouraging the student to learn the material prior to the test (instead of having to “relearn” stuff).
 - It also helps the students develop the ability to assess their own mastery of ideas. Most students go into the test thinking they are generally “ready”, and the “real” grade on the test gives them feedback as to how accurate this personal assessment was and also helps them learn to take “professional” criticisms of their work and benefit from them.

The Bible is clear that we are not expected to be perfect in our Christian walks, but we are to be improving/growing/maturing. This is the same philosophy that we need to bring to our math classes. The goal is not perfection, but learning and improving. Therefore tests should be an opportunity to assess if we have actually learned/improved while not ignoring what we have left to achieve.

Develop the Habits to Encourage your Student

Did we mention that this is really hard work? It is, and both teacher/parent and student will get weary in it sometimes. So work to develop the habits that will help your students be successful without too much drama and help you provoke them to good works.

- For your students
 - Make their daily schedule a habit. Variety can be nice, but we are more efficient with routines.
 - Break up their math into smaller sections throughout the day based on what they should be doing: a read/listen/create study notes time, a work homework problems time, and a short time of reviewing previous study notes.
 - Keep reminding them:
 - We work now to be prepared for whatever plans God has for us in the future, to be profitable servants.
 - God is not so much impressed with our success but our diligence and hard work. He has never made us responsible for our success; He just asks us to be faithful. “Bad” grades

that are a product of our faithful study and best efforts please the Lord. This is our best measure of success.

- The more you practice these “hard” study techniques, the easier it gets and the better we learn; and everyone agrees that math is more fun when we understand what we are doing.
- For you,
 - Expect resistance to deep learning. Determine in advance, that you will not give in. Say out loud (and often, and preferably within hearing distance of your students), that you care too much about them to let them enter adulthood without learning to persevere and do hard things.
 - Our job is not just to ensure that our students have content knowledge but that they have the skills necessary for being profitable in life. This includes perseverance in the face of difficulty. So when your student comes to you with a question on how to do math, make your default answer be “and how can our study notes help us with that?”
Depending on the response you may need to help in different ways:
 - If they are just looking for the “easy” answer, make them go dig into their study notes on their own. You won’t help them until they can tell all the things they tried (looked in their study notes, looked for examples in the book, etc.).
 - If they need help adding a “new” problem to their study notes, you may need to take a few minutes and help them see how the problem is like other problems in their study notes or in the book and have them improve their study notes for the future. If you are not always certain how to help with the process, you may find that it is enough to encourage them in the process.
 - Consider the questions:
 - Which page of your study notes will help you with this problem?
 - At which step in your notes did you get stuck?
 - What should we add to your notes to help you figure this question out in the future? (and make sure they express it in their own words)
 - Explaining a problem in “English” often answers students’ questions. When they come to you with a problem/question, make them explain what the problem is asking in English and then see if they still have a question. Often they’ve answered their own questions, but if they do, at least everyone knows what the problem really is (or at least what the student thinks it is). Sometimes the challenge is for the students to “read” the problem correctly, so they aren’t able to solve it because they didn’t actually understand it.
 - Real learning that sticks for the long haul takes hard work and struggle. We don’t enjoy watching our children struggle, but we can’t steal their opportunity to learn by removing the struggle. So develop the habit of letting them struggle some (even telling them that they need to struggle with ideas, and that’s okay because that’s how they learn).
 - Change how you view mistakes. Get in the habit of using your students’ mistakes to help them refine their understanding. Talk about mistakes as positive learning experiences with your students.

We learn the most from our mistakes in life and in math. Help your students value their mistake as an opportunity to better learn an idea. Help your students build resilience;

learning that our first try won't be perfect (or maybe even good), but that's okay we keep trying is a necessary job skill. We can develop a lot of resilience when studying math (it's easy to be wrong).

- Get in the habit of responding to questions with questions that will help your students figure out their answer "on their own." Consider questions like
 - Can you find a similar example in the book/online?
 - Can you show me the same place in the example that you are getting stuck in your problem?
 - What was the next step in the example? How would it look in your problem?
 - Now that you've figured it out, what should you add to your study notes so that you can remember what you figured out?

These habits will take effort when you start to develop them but will get easier as you practice them, so persevere.

Study Efficiently

Goal: to spend no more time learning something than necessary

There are two ways to learn something: by doing it over and over and over and over again until you "get it" OR by understanding how it works and then practicing just enough to ensure speed and accuracy.

Consider selecting some of the following to work on to develop efficient learning.

- Work on "quality hours" not clock hours – 15 minutes of intense concentration is much more effective than 30 minutes of distracted focus
 - Consider asking your older students to rate their concentration level. You can use a quality hours metric (where *quality hours* = *clock hours* times % of concentration) to help your students measure/improve their efficiency (possibly also as motivator in some way).
 - Help younger students develop ways to improve their concentration. You may wish to use small snacks/stickers/etc. as rewards for improvement. You can also help them arrange their physical space to encourage good focus.
- Develop comfort with a time-pressured environment (the faster we work, the less time we spend).
 - In addition to timed tests, give shorter timed quizzes (to build up comfort with timed exercises in preparation for the timed tests and to improve speed in general)
 - Remember the goal will be to push the students to do just a little bit better than they are doing.
- Let grades reflect actual understanding:
 - Don't let them correct answers after the fact for credit – "learning" material twice (once before and once while correcting) is not efficient. It's actually less time consuming in the long run to spend a little more time before the test and learn the material more fully the first time.
 - Don't give access to study notes/textbooks/test solutions/etc. during the testing period – this may improve the short term test scores, but it means the students will have to relearn



ideas not learned well (for which they needed the additional resources to complete the test answers) in the future.

- Don't let them see solutions until they've tried stuff (find a similar example in the book/online and see if they can mimic it). This forces them to assimilate the information as their own and recreate it on a new problem, as opposed to just copying out the actual solution.
- Don't let them be calculator dependent. Calculators are good tools, but we never want the use of the calculator to slow a student down or to hide the fact that they don't really understand how the problem works. The BJU math faculty have developed a "guiding principles document" on how we use calculators in our math classrooms. You may find the general principles helpful. <https://math.bju.edu/media/bju-math-division/bju-math-department/melissa-gardenghi/BJU-Math-Department-on-Technology.pdf>
- Integrate math work with other skill sets, such as the development of critical reading/writing through math.
 - Have the students write (in "English") about the characteristics of the problems that will use a particular solution technique. You may be able to use it as a writing assignment (even though the topic is math).
 - Have the students read the discussion of the solution technique and make a bulleted list that will tell how to solve any similar problem. You may be able to use it a reading assignment, developing the skill of critical reading – understanding what is said in the selection and the consequences of those statements.
 - Have the students give an oral presentation or a "teaching session" on a mathematical topic they are studying (possibly with the development of some visual aids). You may be able to use it as a speech assignment.
- To complete assigned homework (aka actually mastering ideas, NOT coming up with numerical answers) as efficiently as possible, do some or all of the following before "starting the homework"
 - Test your students' understanding by having them "teach" you how to do a problem
 - Make them explain each step in "normal English" without using "mathy" language
 - Ask them, how was I supposed to know to do it that way?
 - Pick a problem to work, and only do exactly what they tell you to do

If they can do these smoothly, then the problems should progress quickly (if they are paying attention).

- Use homework problems strategically. Remember you can either learn by blind repetition (which can take forever) or by purposeful learning with enough repetition to be confident, fast, and accurate.
 - If your student has "taught" you their math lesson, then assign just enough homework so that they are fast and accurate
 - Perhaps require them to complete ___ of each "type" of homework problem within a ___ minute time period correctly before they can be done

Now you are rewarding understanding, problem mastery, and computational speed/accuracy, NOT task completion